Noise

In electromagnetic systems, the energy per photon = hν.
In communication systems, noise can be either quantum or additive from the measurement system (receiver, etc).

The additive noise power is 4kTB,

k is the Boltzman constant
T is the absolute temperature
B is the bandwidth of the system.

When making a measurement (e.g. measuring voltage in a receiver), noise energy per unit time 1/B can be written as 4kT.

$$SNR = \frac{Nh\nu}{\sqrt{N}h\nu + AdditiveNoise} = \frac{Nh\nu}{\sqrt{N}h\nu + 4kT}$$

$\sqrt{N}$ comes from the standard deviation of the number of photons per time element.

SNR in x-ray systems

$$SNR = \frac{Nh\nu}{\sqrt{N}h\nu + AdditiveNoise} = \frac{Nh\nu}{\sqrt{N}h\nu + 4kT}$$

When the frequency ν<< GHz, $4kT >> h\nu$

In the X-ray region where frequencies are on the order of 10^{19}:

$$h\nu >> 4kT$$

X-ray is quantum limited due to the discrete number of photons per pixel.

We need to know the mean and variance of the random process that generate x-ray photons, absorb them, and record them.

Recall: $h = 6.63 \times 10^{-34}$ Js

$k = 1.38 \times 10^{-23}$ J/K

Discrete-Quantum Nature of EM radiation detection

- Detector does not continuesly absorb energy
- But, absorb energy in increments of $h\nu$
- Therefore, the output of detector cannot be smooth
- But also exhibit Fluctuations known as quantum noise, or Poisson noise (as definition of Poisson distribution, as we see later)

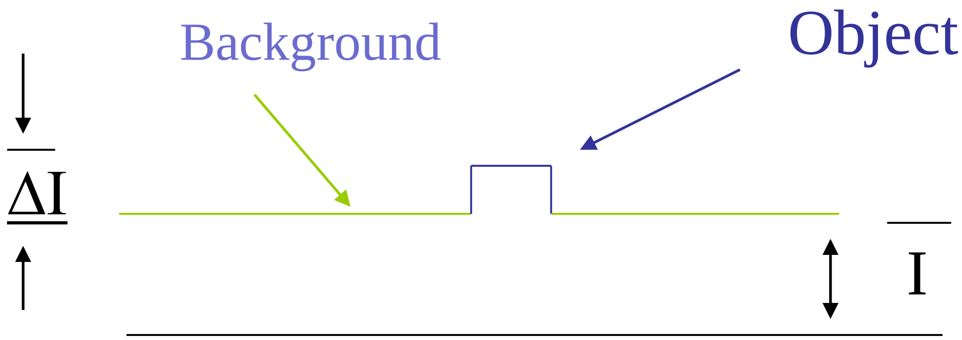
Noise in x-ray system

- $h\nu$ is so large for x-rays due to necessity of radiation dose to patient, therefore:
  - 1) Small number of quanta is probable to be detected
  - 2) Large number of photons is required for proper density on Film
    - 10^7 x-ray photons/cm^2 exposed on Screen
    - 10^{11}-10^{12} optical photons/cm^2 exposed on Film
- Therefore, with so few number of detected quanta, the quantum noise (poisson fluctuation) is dominant in radiographic images

Assumptions

- Stationary statistics for a constant source and fixed source-detector geometry
- Ideal detector which responds to every phonon impinging on it

Motivation:



$$\text{Contrast} = \Delta I / \bar{I}$$

$$\text{SNR} = \Delta I / \sigma_I = C\bar{I} / \sigma_I$$

We are concern to detect some objects (here shown in blue) that has a different property, eg. “attenuation”, from the background (green).

To do so:

we have to be able to describe the random processes that will cause the x-ray intensity to vary across the background.

binomial distribution:

is the discrete probability distribution of the number of successes (eg. Photon detection) in a sequence of *n* independent experiments (# of interacting photons).

Each photon detection yields success with probability *p*.

If experiment has only 2 possible outcomes for each trial (eg. Yes/No), we call it a Bernouli random variable.

Success: Probability of one is p

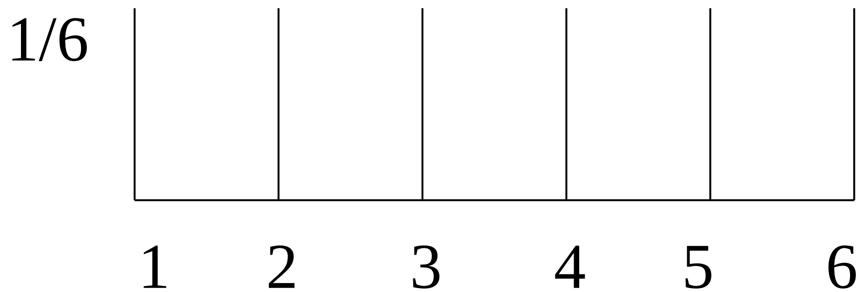
Failure: Probability of the other is 1 - p

Probability

Rolling dies

The outcome of rolling the die is a random variable of discrete values.

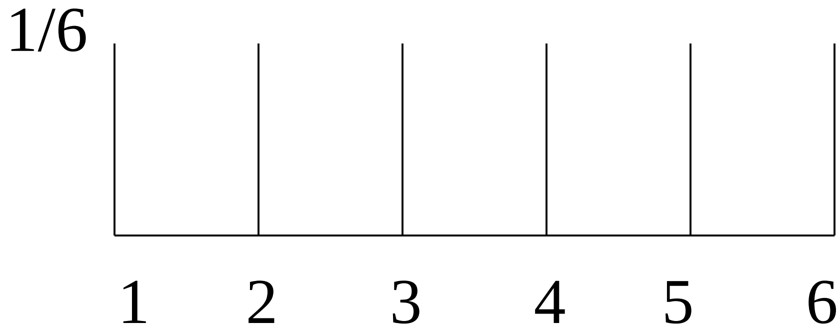
Let's call this random variable X. We write then that the probability of X being value n (eg. 2) is $p_x(n) = 1/6$



Note: Because the probability of all events is equal, we refer to this event as having a uniform probability distribution

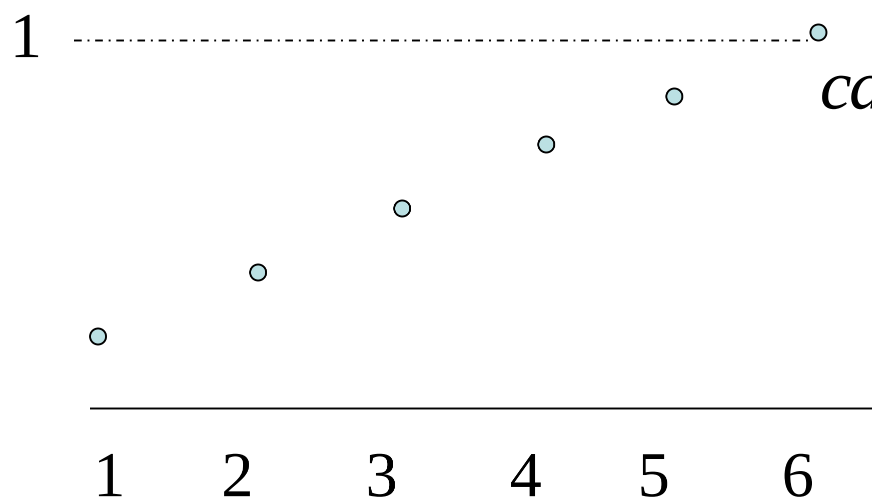
Probability

Probability Density Function (pdf)



$$p_X(j) = \frac{1}{6}$$

Cumulative Density Function



$$cdf(m) = F_x(x) = \sum_{j=1}^{j=m} p_X(j)$$

pdf is derivative of cdf:

$$p_X(x) = dF_X(x)/dx$$

$$0 \le F_X(x) \le 1$$

Zeroth Order Statistics

- Not concerned with relationship between events along a random process
- Just looks at one point in time or space
- Mean of X, μ_X, or Expected Value of X, E[X]
  - Measures first moment of $p_X(x)$

$$\mu_X = \int_{-\infty}^{\infty} x p_X(x) dx$$

- Variance of X, σ^2_X , or $E[(X-\mu)^2]$
  - Measure second moment of $p_X(x)$

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p_X(x) dx$$

Standard deviation

$$\sigma_X = std$$

Zeroth Order Statistics

- Recall E[X] $= \int_{-\infty}^{\infty} x p_X(x)dx$
- Variance of X or E[(X-μ)^2]

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p_X(x)dx$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} x^2 p_X(x)dx - 2\mu \int_{-\infty}^{\infty} x \ \ p_X(x)dx + \mu^2 \int_{-\infty}^{\infty} p_X(x)dx$$

$$\sigma_X^2 = E[X^2] - 2\mu E[X] + \mu^2$$

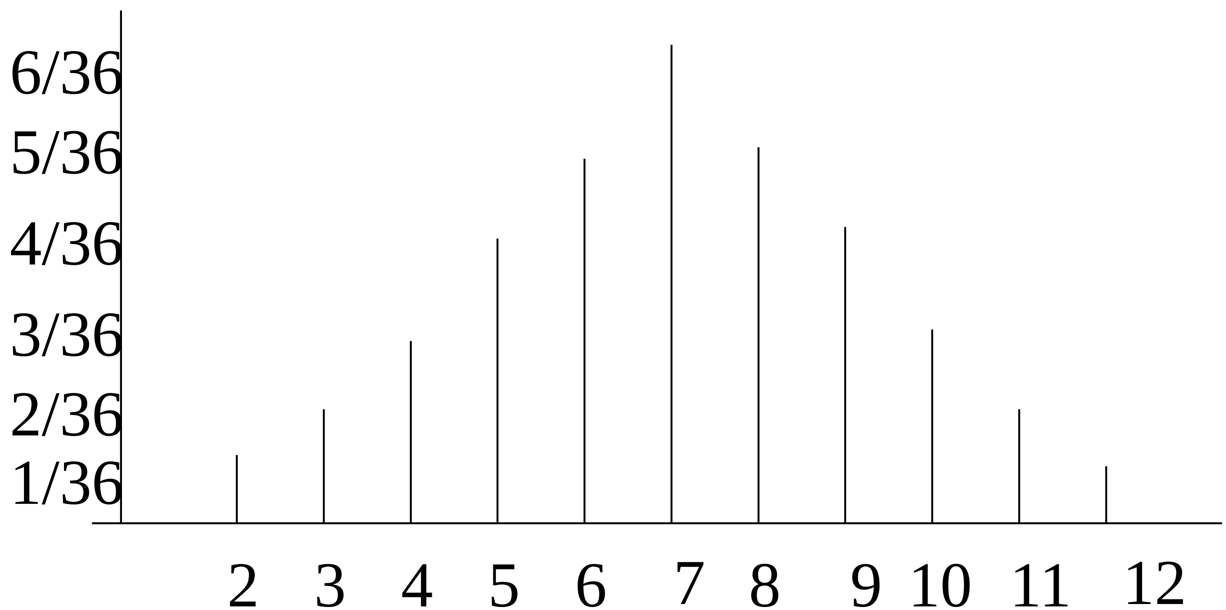
$$\sigma_X^2 = E[X^2] - E^2[X] = E[X^2] - \mu^2$$

Binomial Random Variable

p(j) for throwing 2 die is 1/36:

Let die 1 experiment result be x and called Random Variable X
Let die 2 experiment result be y and called Random Variable Y
With independence: $p_{XY}(x,y) = p_X(x)\, p_Y(y)$

$E[xy] = \int\int xy\, p_{XY}(x,y)\, dx\, dy = \int x\, p_X(x)\, dx \int y\, p_Y(y)\, dy = E[X]\, E[Y]$



Other helpful Theorems

$E[X+Y] = E[X] + E[Y]$ Always

$E[aX] = aE[X]$ Always

$\sigma^2_x = E[X^2] - E^2[X]$ Always

$\sigma^2(aX) = a^2 \sigma^2_x$ Always

$E[X + c] = E[X] + c$

$Var(X + Y) = Var(X) + Var(Y)$ only if the X and Y are statistically independent.

For n trials,

P[X = i] is the probability of i successes in the n trials
X is said to be a binomial variable with parameters (n,p)

$$p[X = i] = \frac{n!}{(n-i)!i!} p^i (1-p)^{n-i}$$

Example

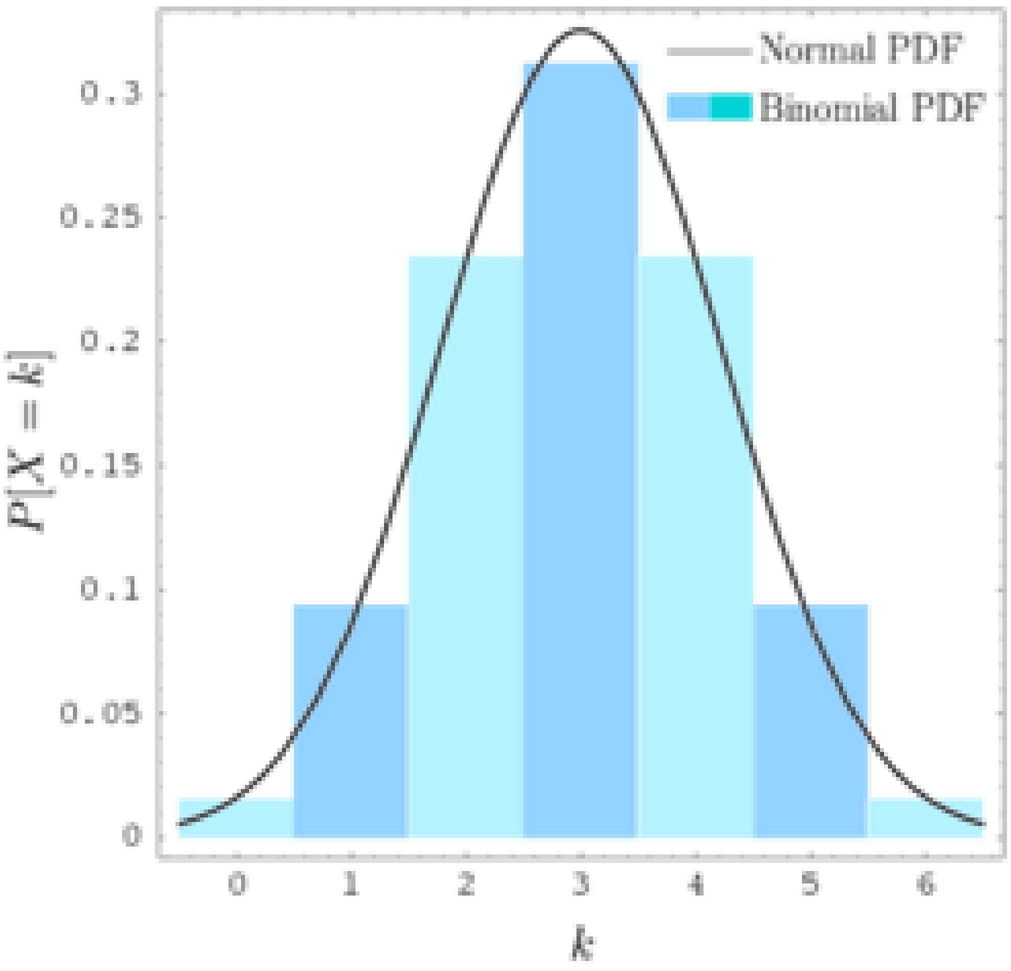
Roll a die 10 times (n=10).
In this game, you win if you roll a 6.
Anything else - you lose

What is P[X = 2], the probability you win twice (i=2)?

$$p[X = i] = \frac{n!}{(n-i)!i!} p^i (1-p)^{n-i}$$

= (10! / 8! 2!) (1/6)2 (5/6)8

= (90 / 2) (1/36) (5/6)8 = 0.2907



Binomial PDF and normal approximation for $n = 6$ and $p = 0.5$.

Limits of binomial distributions

- As n approaches ∞ and p approaches 0, then the Binomial(n, p) distribution approaches the Poisson distribution with expected value $\lambda = np$.

- As n approaches ∞ while p remains fixed, this distribution approaches the normal distribution with expected value 0 and variance 1

- (this is just a specific case of the Central Limit Theorem).

Recall: If p is small and n large so that np is moderate, then an approximate (very good) probability is:

$P[X=i] = e^{-\lambda} \lambda^{i} / i!$ Where $np = \lambda$

the probability exactly i events happen

Poisson Random Variable

With Poisson random variables, their mean is equal to their variance!

$E[X] = \sigma_x^2 = \lambda$

Example

Let the probability that a letter on a page is misprinted is 1/1600. Let's assume 800 characters per page. Find the probability of 1 error on the page.

Using Binomial Random Variable Calculation:

$i = 1$, $p = 1/1600$ and $n = 800$

$P[X = 1] = (800! / 799!)(1/1600)(1599/1600)^{799}$

Very difficult to calculate some of the above terms.

But using Poisson calculation:

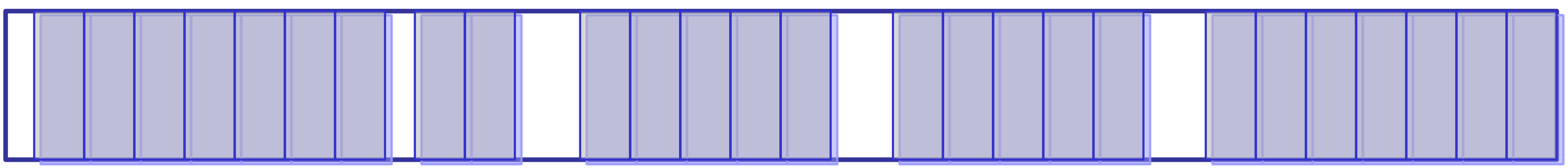
$P[X = i] = e^{-\lambda} \lambda^{i} / i!$ Here, so $\lambda = np = ½$

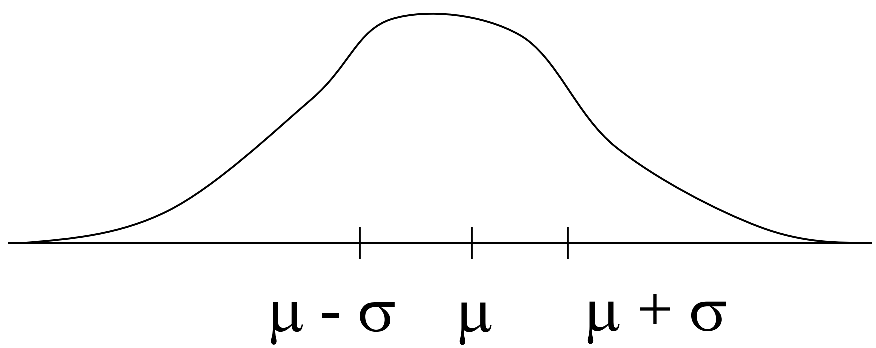
So $P[X=1] = 1/2\, e^{-0.5} = .30$

Other Poisson Random Variables

1) Number of biscuits sold in a store each day

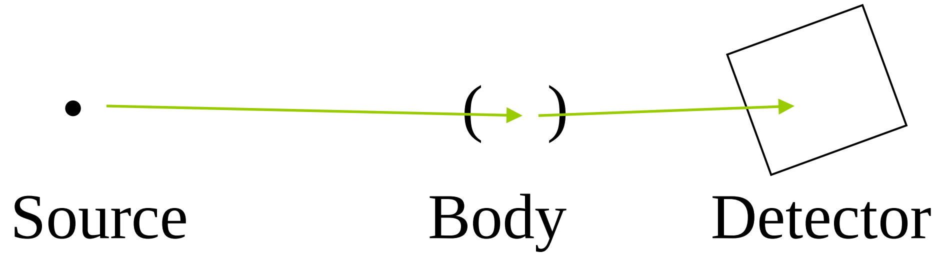
2) Number of x-rays discharged off an anode





$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} \right]$$

To find the probability density function that describes the number of photons striking on the Detector pixel



1) Probability of X-ray emission is a Poisson process:

$$P\left(\frac{i_{emissions}}{unit-time}\right) = N_0^i \frac{e^{-N_0}}{i!}$$

N_0 is the average number of emitted X-ray photons (i.e λ in the Poisson process).

2) Transmission -- Binomial Process

transmitted $p = e^{-\int u(z)\, dz}$

interacting $q = 1 - p$

3) Cascade of a Poisson and Binary Process still has a Poisson Probability Density Function

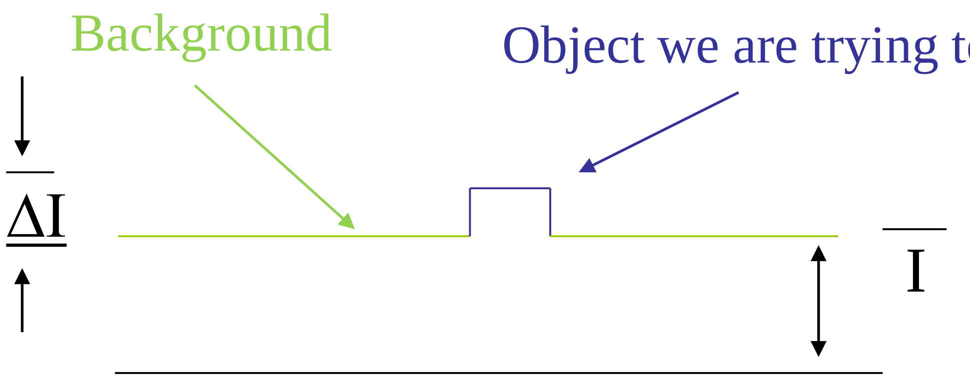
- Q(i) represents transmission of Emitted photons:

$$Q(i) = (pN_0)^i \frac{e^{-pN_0}}{i!}$$

With Average Transmission: $\lambda = pN_0$

Variance: $\sigma^2 = pN_0$

SNR



$$Contrast = \frac{\Delta I}{\bar{I}}$$

$$SNR = \frac{\Delta I}{\sigma_I} = \frac{C\bar{I}}{\sigma_I}$$

SNR Based on the number of photons (N):

then ΔN describes the signal :

$$SNR = \frac{\Delta N}{\sigma_N} = \frac{\Delta N}{\sqrt{N}} = \frac{CN}{\sqrt{N}} = C\sqrt{N} \qquad where: \qquad C = \frac{\Delta N}{N}$$

SNR based on Detected Photons per Pixel

The average number of photons N striking a detector depends on:

1- Source Output (Exposure), Roentgen (R) (Considering Geometric efficiency $\Omega/4\pi$ (fractional solid angle subtended by the detector)

2- Photon Fluence/Roentgen $$\frac{\Phi}{R} = 2.5 \bullet 10^{10} \frac{Photons/cm^2}{R}$$ for moderate evergy

3- Pixel Area (cm^2)

4- Transmission probability p

$$N = \Phi A R \exp[- \int \mu dz]$$

Let $t = \exp[-\int \mu \, dz]$ and Add a recorder with quantum efficiency η

$$SNR = C\sqrt{N} = C\sqrt{\eta \Phi A R t}$$

Example chest x-ray:

50 mRad= 50 mRoentgen

$\eta = 0.25$

Res = 1 mm

t = 0.05

What is the SNR as a function of C?

$$SNR = C\sqrt{N} = C\sqrt{\eta N}$$

SNR based on Light Photons per Pixel on Film

Consider the detector

M $\rightarrow$ X light photons / capture $\rightarrow$ Y light photons

Captured Photons
In Screen (Poisson)

What are the zeroth order statistics on Y?

$$Y = \sum_{m=1}^{M} X_m$$

Y depends on the number of x-ray photons M that hit the screen, a Poisson process.

Every photon that hits the screen creates a random number of light photons, also a Poisson process.

What is the mean of Y? (This will give us the signal level in terms of light photons)

$$Y = \sum_{m=1}^{M} X_m$$

Mean

$$E[Y] = E[\sum_{m=1}^{M} X_m]$$

Expectation of a Sum is Sum of Expectations (Always).
There will be M terms in sum.

$$E[Y] = E[X_1] + E[X_2] + ...E[X_m]$$

Each Random Variable X has same mean.
There will be M terms in the sum

E [Y] = E [M] E [X]   Sum of random variables

E [M] = η N   captured x-ray photons / element

E [X] = g_1 mean # light photons/single x-ray capture

so the mean number of light photons is $E[Y] = \eta N g_1$.

What is the variance of Y? (This will give us the std deviation)

$$Y = \sum_{m=1}^{M} X_m$$

We consider the variance in Y as a sum of two variances:

1. The first will be an uncertainty in M, the number of incident X-ray photons.

2. The second will be due to the uncertainty in the number of light photons generated per each X-ray photon, X_m.

What is the variance of Y due to M? $$Y = \sum_{m=1}^{M} X_m$$

Considering M (x-ray photons) as the only random variable and X (Light/photons) as a constant,

then the summation would simply be: Y = MX.

The variance of Y is: $\sigma_y^2 = X^2 \sigma_M^2$

(Recall that multiplying a random variable by a constant increases its variance by the square of the constant. Note: The variance of M effects X)

But X is actually a Random variable, so we will write X as E[X]

Therefore, Uncertainty due to M is: $\sigma_{y1}^2 = [E[X]]^2 \sigma_M^2$

What is the variance of Y do to X (Light/photons)? $Y = \sum_{m=1}^{M} X_m$

Here, we consider each X in the sum as a random variable but M is considered fixed:

Then the variance of the sum of M random variables would simply be $M.\sigma_x^2$

Note: Considering that the variance of X has no effect on M (ie. each process that makes light photons by hitting a x-ray photon is independent of each other)

Therefore, Uncertainty due to X is: $\sigma_{y2}^2 = E[M]\,\sigma_X^2$

$\sigma_M^2 = E[M] = \eta N$ Recall M is a Poisson Process

$\sigma_X^2 = E[X] = g_1$ Generating light photons is also Poisson

$$\sigma_Y^2 = \sigma_{y1}^2 + \sigma_{y2}^2 = [E[X]]^2 \sigma_M^2 + E[M]\,\sigma_X^2 = \eta N g_1^2 + \eta N g_1$$

Uncertainty of Y due to X

Uncertainty of Y due to M

$$SNR = \frac{CE[Y]}{\sigma_y} = \frac{C\eta N g_1}{\sqrt{\eta N g_1 + \eta N g_1^2}} = \frac{C\eta N g_1}{\sqrt{\eta N g_1}\sqrt{1+g_1}}$$

Dividing numerator and denominator by g_1

$$= \frac{C\sqrt{\eta N}}{\sqrt{1+\frac{1}{g_1}}}$$

What can we expect for the limit of g_1, the generation rate of light photons?

$$g_1 = \frac{h\nu_{x-ray}}{h\nu_{light}} = \frac{\lambda_{light}}{\lambda_{x-ray}} = \frac{5000A^\circ}{.25A^\circ} \approx 20{,}000$$

Actually, half of photons escape and energy efficiency rate of screen is only 5%. This gives us a $g_1 = 500$

Since $g_1 >> 1$, $SNR \approx C\sqrt{\eta N}$

SNR based on Density grains per Pixel on Film

We still must generate pixel grains

$$W = \sum_{m=1}^{Y} Z_m$$ where W is the number of silver grains developed

Y → [Z] → W grains / pixel

Light Photons / pixel

Z = developed Silver grains / light photons

Let $E[Z] = g_2$, the number of light photons to develop one grain of film.

Then, $\sigma_z^2 = g_2$ (since this is a Poisson process, i.e. the mean is the variance).

$E[W] = E[Y]\,E[Z] = \eta N g_1 \cdot g_2$

Recall: $\sigma_Y^2 = \eta N g_1^2 + \eta N g_1$

$E[Z] = \sigma_z^2 = g_2$ Number of light photons needed to develop a grain of film

$\sigma_W^2 = \sigma_Y^2 E^2[Z] + E[Y]\,\sigma_z^2$

uncertainty in light photons

uncertainty in gain factor z

$$\sigma_w^2 = (\eta N g_1^2 + \eta N g_1) g_2^2 + g_1 \eta N g_2$$

$$\sigma_w = g_1 g_2 \sqrt{\eta N} \sqrt{1 + 1/g_1 + 1/g_1 g_2}$$

$$SNR = \frac{CE[W]}{\sigma_W} = \frac{C\sqrt{\eta N}}{\sqrt{1 + 1/g_1 + 1/g_1 g_2}}$$

Recall $g_1 = 500$ (light photons per X-ray)

$g_2 = 1/200$ light photon to develop a grain of film

That is one grain of film requires 200 light photons.

$1/g_1 << 1$

$$SNR = \frac{C\sqrt{\eta N}}{\sqrt{1 + 1/500 + \frac{\frac{1}{500}}{200}}}$$

$$SNR \approx 0.85C\sqrt{\eta N}$$

Putting it all together…

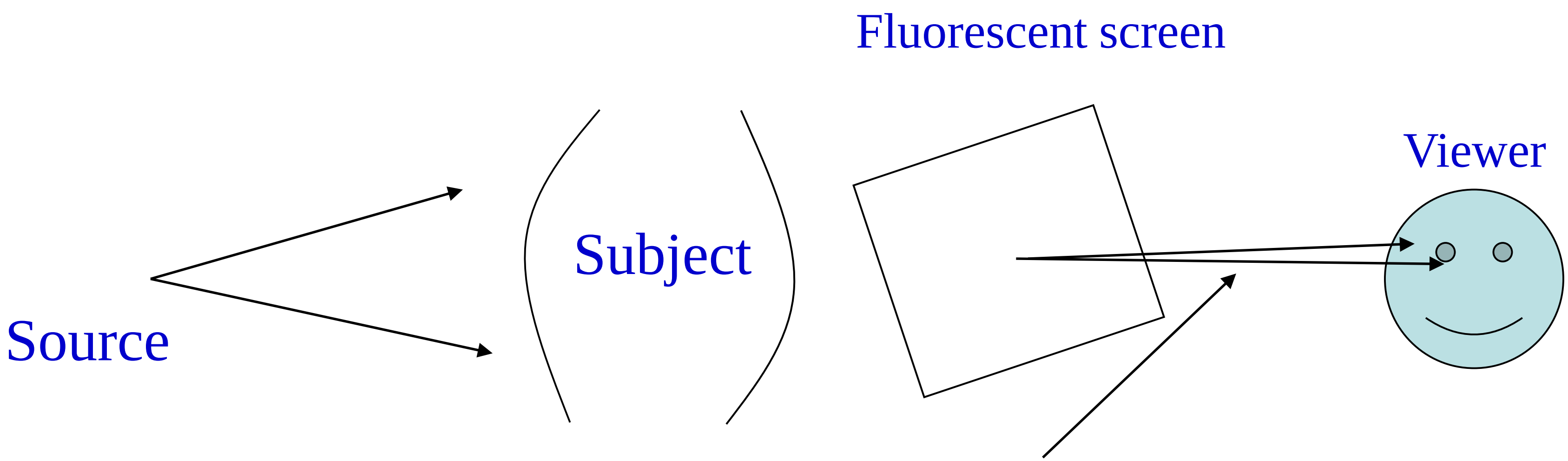
M →	X →	Z →	W
Transmitted photons	g_1 light photons /x-ray	g_2 grains/ light photon	developed grains

For N as the average number of transmitted, not captured, photons per unit area.

$$SNR = \frac{C\sqrt{\eta N}}{\sqrt{1 + 1/g_1 + 1/g_1 g_2}}$$

Fluoroscopy

-Old Method

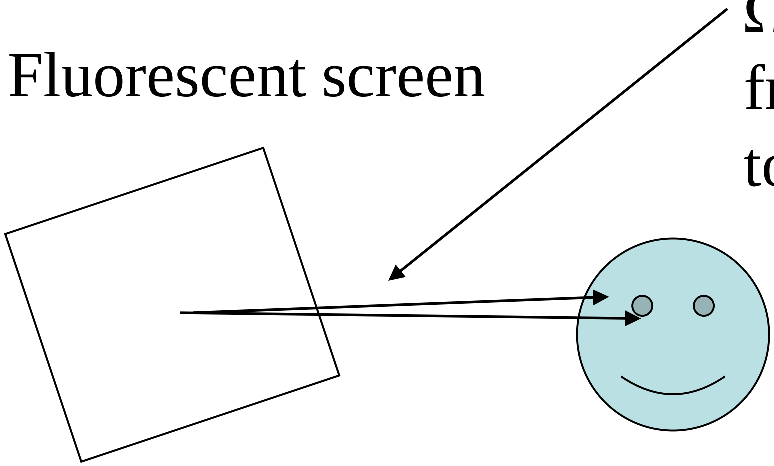


Ω is the solid angle subtended
from a point on the detector
to the pupil

If a fluorescent screen is used instead of film, the eye will only capture a portion of the light rays generated by the screen.
How could the eye's efficiency be increased?

Fluorescent screen

Ω is the solid angle subtended from a point on the detector to the pupil



Let's calculate the eye's efficiency capturing light η_ε

$$\eta_\varepsilon = \frac{\Omega}{4\pi} T_e = \frac{A}{4\pi r^2} T_e$$

r = viewing distance (minimum 20 cm)

$T_e \equiv$ retina efficiency (approx. 0.1)

A = pupil area ≈ 0.5 cm^2 (8 mm pupil diameter)

Recall

$$SNR = \frac{C\sqrt{\eta N}}{\sqrt{1 + 1/g_1 + 1/g_1 g_2}}$$

In fluoroscopy:

$g_1 = 10^3$ light photons / x-ray

$g_2 = \eta_\varepsilon$

$\eta\varepsilon \approx 10^{-5}$ (at best) Typically $\eta\varepsilon \approx 10^{-7}$

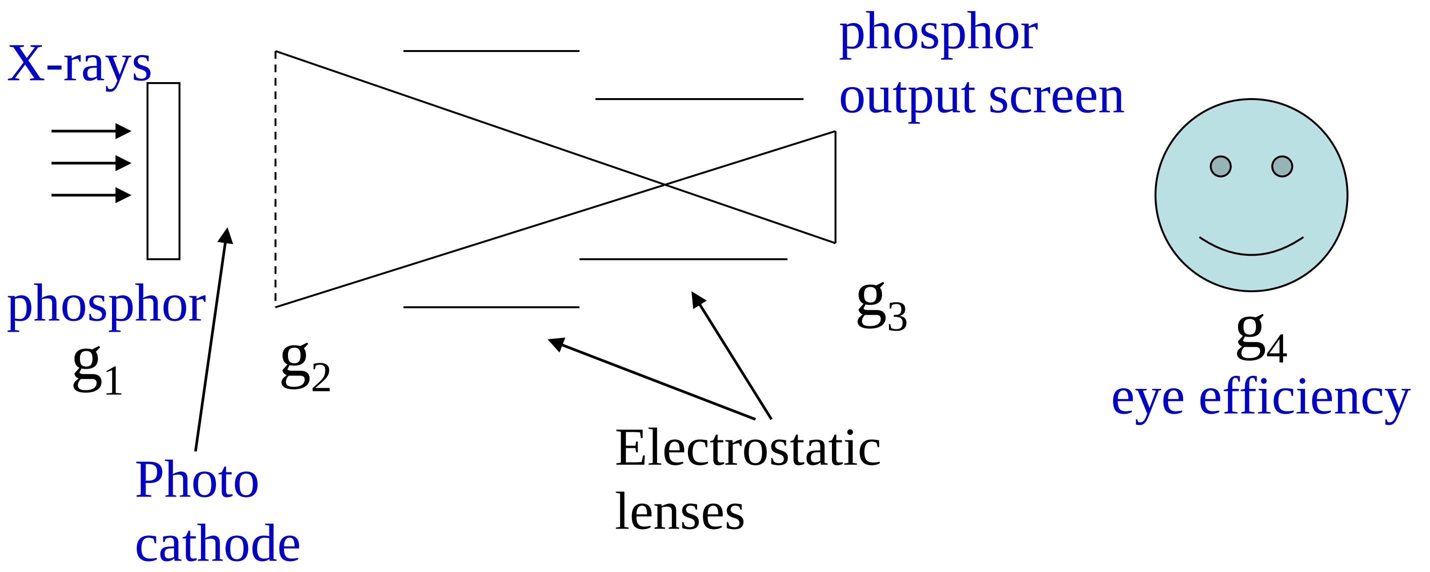
$g_1 g_2 = 10^3 10^{-5} = 10^{-2}$ at best

Therefore loss in SNR is about 10

We have to up the dose by a factor of 100! (or, more likely, to compromise resolution rather than dose)

At each stage, we want to keep the gain product >> 1 or quantum effects will harm SNR.

Image Intensifier



g_1 = Conversion of x-ray photon to Light photons in Phosphor
g_2 = Conversion of Light photons into electrons
g_3 = Conversion of accelerated electron into light photons

$$SNR = \frac{C\sqrt{\eta N}}{\sqrt{1+\frac{1}{g_1}+\frac{1}{g_1 g_2}+\frac{1}{g_1 g_2 g_3}+\frac{1}{g_1 g_2 g_3 g_4}}}$$

$g_1 = 10^3$ light photons /captured x-ray

g_2 = Electrons / light photons = 0.1

$g_1 g_2 = 100$

g_3 = emitted light photons / electron = 10^3

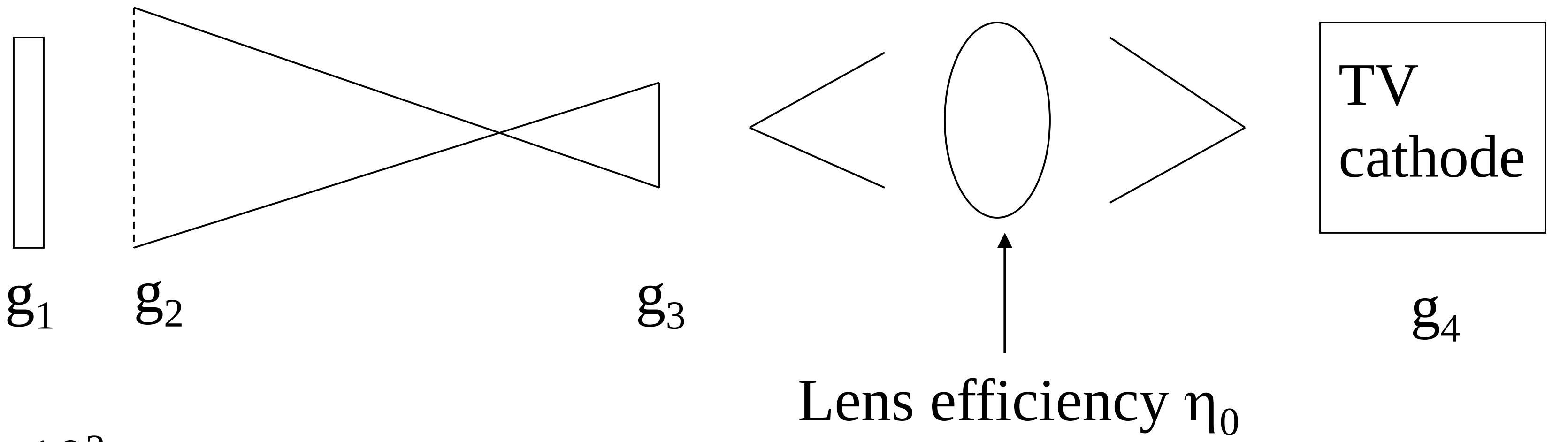
$g_4 = \eta_\varepsilon$ eye efficiency = 10^{-5} optimum

$g_1 g_2 g_3 g_4 = 10^3\, 10^{-1}\, 10^3\, 10^{-5} = 1$

$$SNR = \frac{C\sqrt{\eta N}}{\sqrt{1 + \frac{1}{g_1} + \frac{1}{g_1 g_2} + \frac{1}{g_1 g_2 g_3} + \frac{1}{g_1 g_2 g_3 g_4}}}$$

$$\text{SNR loss} = \frac{1}{\sqrt{1+1}} = \sqrt{2}/2$$

Add a TV



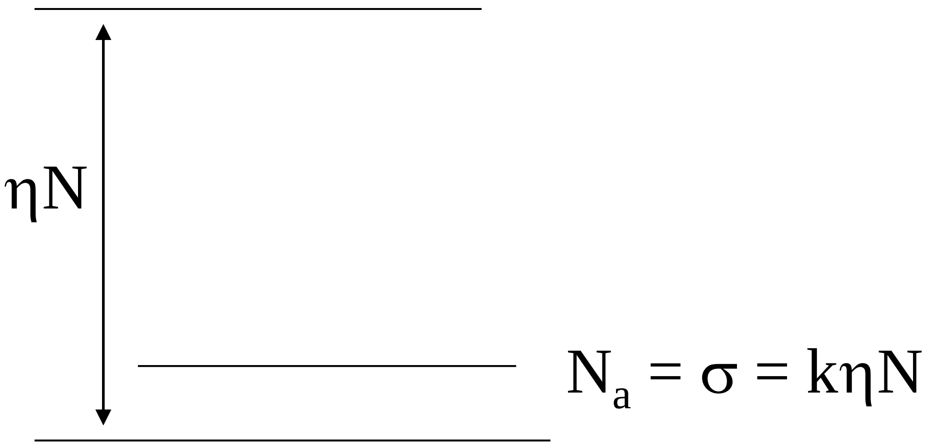
$g_1 = 10^3$
$g_2 = 10^{-1}$
$g_3 = 10^3\,\eta_0$
$\eta_0 \approx 0.04$ (Lens efficiency. Much better than eye)
$g_4 = 0.1$ electrons / light photon
$g_1 g_2 g_3 g_4 = 4 \times 10^2$
$g_1 g_2 g_3 g_4 \gg 1$
and all the intermediate gain products >> 1

$$SNR = C\sqrt{\eta N}$$

Additive TV Noise

But TV has an additive electrical noise component.
Let's say the noise power (variance) is N_a^2.

$$SNR = \frac{C\eta N}{\sqrt{\eta N + N_a^2}}$$



-In X-ray, the number of photons is modeled as our source of signal.
- We can consider N_a (which is actually a voltage), as its equivalent number of photons.
- Electrical noise then occupies some fraction of the signal's dynamic range. Let's use k to represent the portion of the dynamic range that is occupied by additive noise.

$$SNR = \frac{C\eta N}{\sqrt{\eta N + k^2\eta^2 N^2}} = \frac{C\eta N}{\eta N\sqrt{\frac{1}{\eta N} + k^2}} = \frac{C}{k}\sqrt{\frac{1}{1 + 1/k^2\eta N}}$$

Typical TV

$$SNR = \frac{C}{k}\sqrt{\frac{1}{1+1/k^2\eta N}}$$

k = 10^{-2} to 10^{-3}

$\eta N = 10^5$ photons / pixel

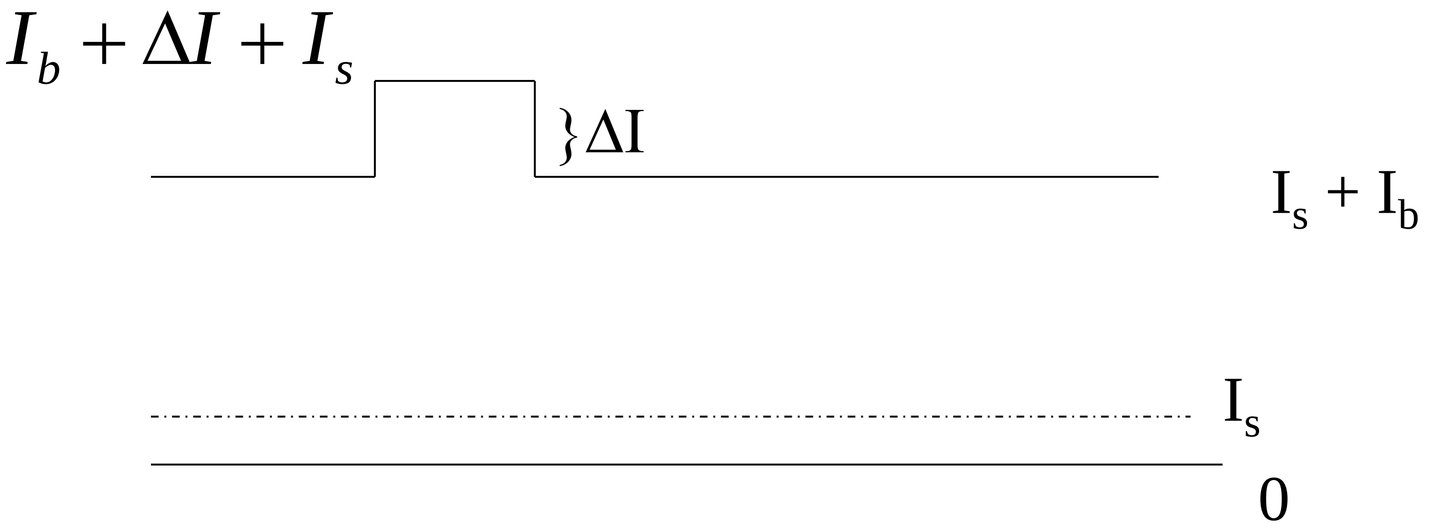
k = 10^{-2} $\quad$ $k^2\eta N = 10$

k = 10^{-3} $\quad$ $k^2\eta N = 10^{-1}$ $\quad$ Much Better!

If $k^2\eta N \ll 1$ $\quad$ $SNR = C\sqrt{\eta N}$

If $k^2\eta N \gg 1$ $\quad$ SNR $\approx$ C/k $\quad$ poor, not making use of radiation

Scatter Radiation



Scatter increases the background intensity.
Scatter increases the level of the lesion.
Let the ratio of scattered photons to desired photons be

$$\psi = \frac{I_s}{I_b}$$

$$C_s = \frac{(I_b + \Delta I + I_s) - (I_s + I_b)}{I_s + I_b}$$

$$C_s = \frac{\Delta I}{I_s + I_b} = \frac{\Delta I}{I_b(1 + \frac{I_s}{I_b})} = \frac{C}{(1 + \frac{I_s}{I_b})}$$

$$\text{Let } \psi = \frac{I_s}{I_b} \text{ then } C_s = \frac{C}{1 + \psi}$$

SNR Effects of Scatter

The variance of the background depends on the variance of trans-mitted and scattered photons.
Both are Poisson and independent so we can sum the variances.

$\sigma_N^2 = \eta N + \eta N_s$ where N_s is mean number of scattered photons

$$SNR = \frac{C\eta N}{\sqrt{\eta N + \eta N_s}} = C\frac{\sqrt{\eta N}}{\sqrt{1+\frac{N_s}{N}}} = C\frac{\sqrt{\eta N}}{\sqrt{1+\Psi}}$$

Here C is the scatter free contrast.

Filtering of Noisy Images

- Imaging system is combination of Linear filters with in turn effects on Noisy signals
- Noise can be Temporal or Spatial in an image
- This can also be classified as Stationary or nonstationary
- If the Random fluctuating input to a system with impulse response of p(t) is $w_{in}(t)$, what can be the mean $\langle w_{out} \rangle$ and variance σ_{out} of the output (which is noise variation):