Extended (Finite) Sources

Collimated X-ray

$$I_d (x_d,y_d) = I_0 \exp [- \int \mu_o (x,y,z)\, dz]$$

X-Ray with a Point Source

$$I_d (x_d,y_d) = I_i \exp [- \sqrt{1+\frac{r_d^2}{d^2}} \int \mu_o ((x_d/d)z, (y_d/d)z, z)\, dz]$$

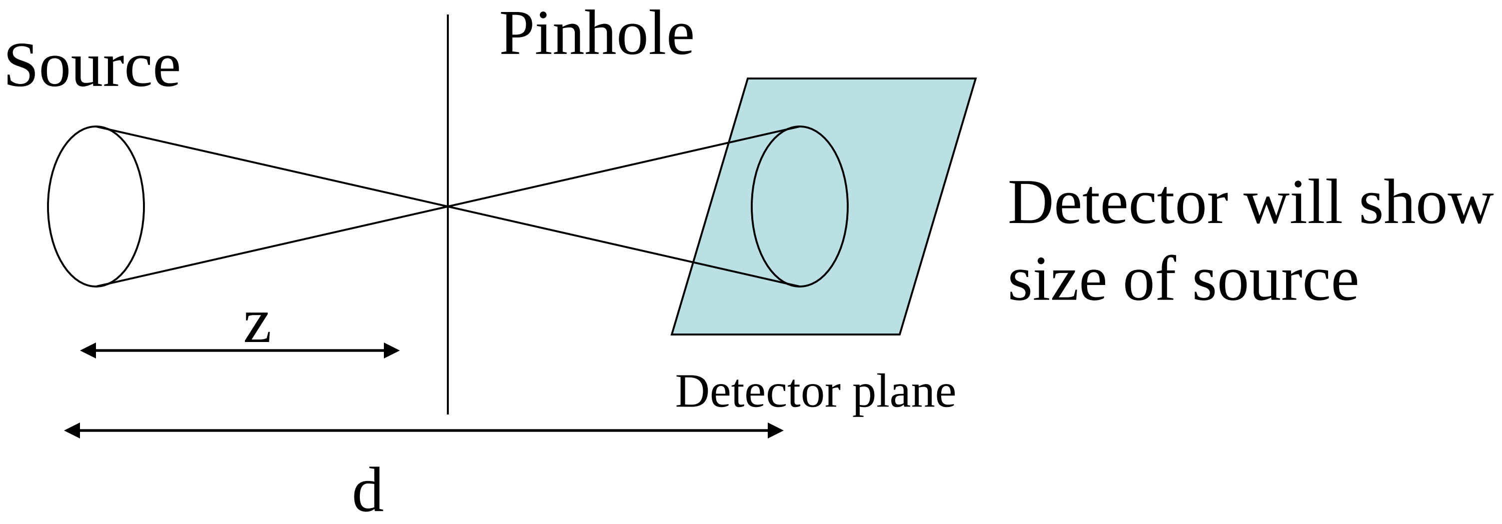
$$I_d (x_d,y_d) = I_i \exp [- \sqrt{1+\frac{r_d^2}{d^2}} \int \mu_o (x_d/M, (y_d/M, z)\, dz]$$

where $M = d/z$

and $I_i = I_o / (1 + (r_d^2/d)^2)^{3/2}$

Finite source

Measuring Source Size



Place pinhole between source and detector. This pinhole Reproduce an inverted image of source magnified by (d-z)/z=m.

The point response $h(x_d,y_d)$ for the pinhole for a source distribution $s(x_s,y_s)$ is:

$$h(x_d, y_d) = K_s s(x_s, y_s) = K_s s(\frac{x_d}{m}, \frac{y_d}{m})$$

Finite source

The total detected Intensity (image) of a transparent object (hole) having transmission $t(x,y) = \exp[-\mu(x,y)\,\delta(z - z_o)]$ imaged by a finite x-ray source, $s(x,y)$ is obtained by convolution process:

$$I_d(x_d, y_d) = Kt\left(\frac{x_d}{M}, \frac{y_d}{M}\right) ** s\left(\frac{x_d}{m}, \frac{y_d}{m}\right)$$

$$where: \qquad M = \frac{d}{z}\ and\ m = \frac{-(d-z)}{z}$$

The detected image will be the convolution of a Magnified object and a magnified source.

In frequency domain:

$$I_d(u,v) = KM^2 m^2 T(Mu, Mv) S(mu, mv)$$

Geometric Ray Optics

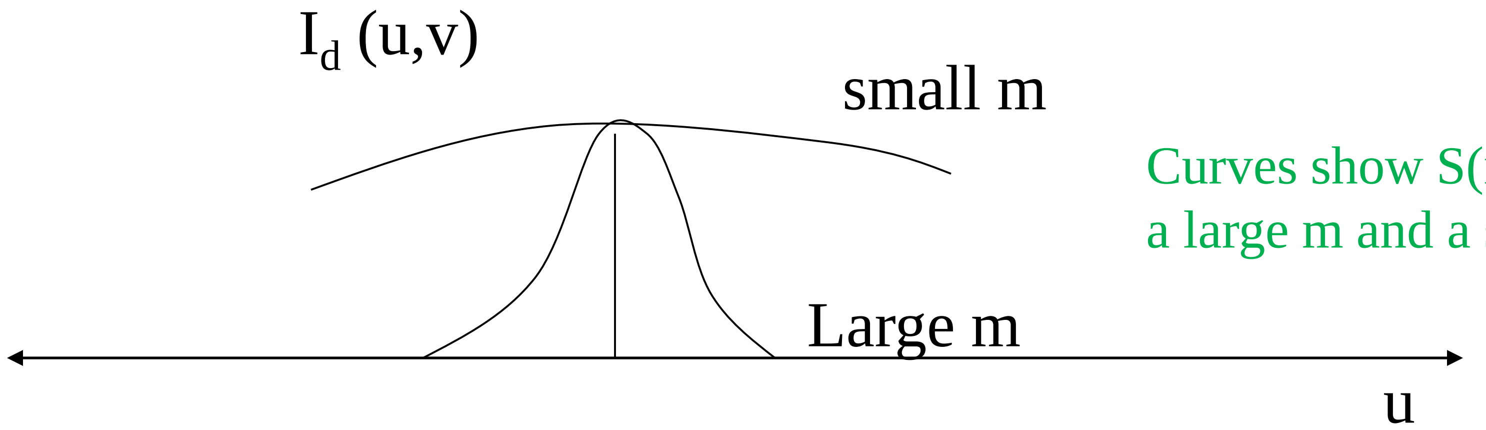
$$I_d(x_d, y_d) = Kt\left(\frac{x_d}{M}, \frac{y_d}{M}\right) ** s\left(\frac{x_d}{m}, \frac{y_d}{m}\right)$$

In the Fourier domain:

$I_d\,(u,v) = KM^2m^2\, T\,(Mu, Mv)\; S\,(mu,mv)$

Object is magnified by $M=d/z$

Source magnification causes distortion: $m = \dfrac{-(d-z)}{z}$



Curves show S(mu,mv) for a large m and a small m

Detailed Analysis

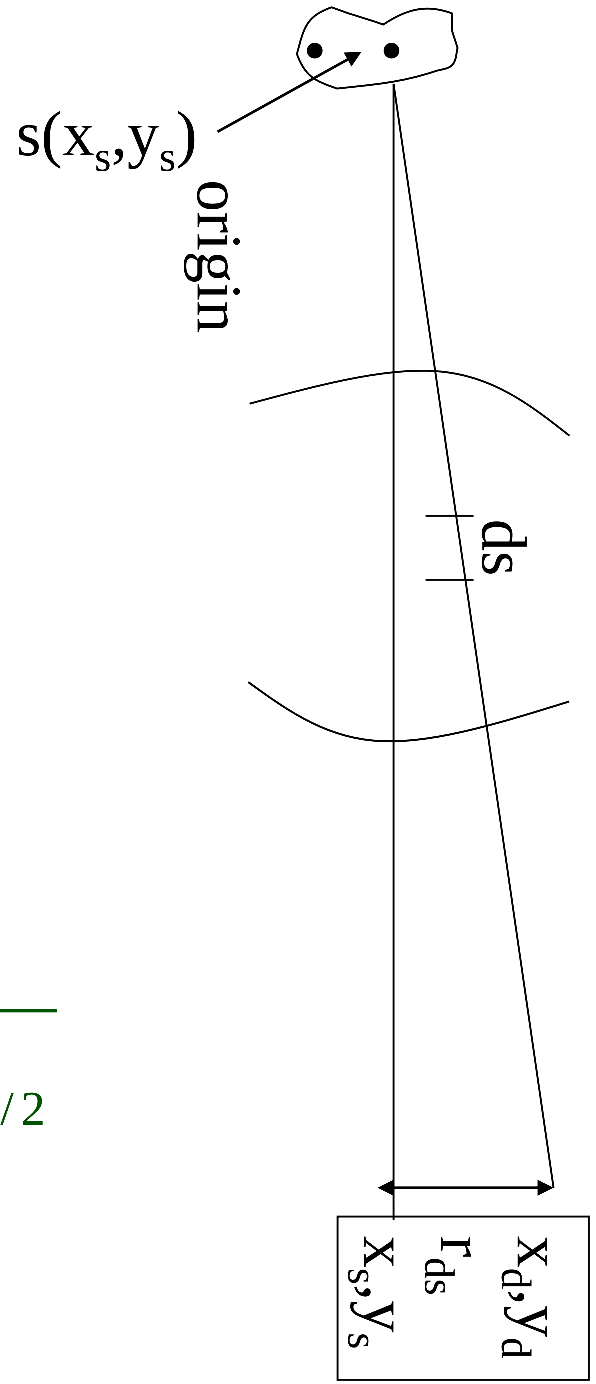
1) First, no object. Differential Intensity at detector plane:

$$dI_i\,(x_d,\, y_d) = dI_o \cos^3\theta$$

$$dI_o = s(x_s,y_s)\; dx_s\; dy_s/(4\pi d^2)$$

Source units $((N/mm^2)/min)$

$$dI_d(x_d, y_d) = dI_0 \frac{1}{(1+\frac{r_d^2}{d^2})^{3/2}}$$



2) Now let's put in an object (for object obliquity). What is the output of x_d, y_d given a source at x_s, y_s?

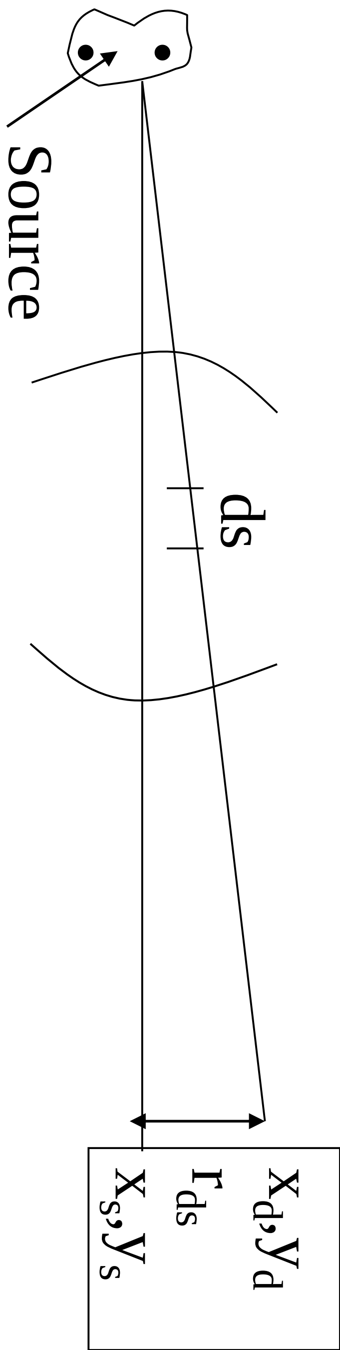
We call this the differential detected image.

$$dI_d\ (x_d, y_d, x_s, y_s) = dI_i \exp\left[-\int \mu\ (x,y,z)\ ds\right]$$

As before, we will evaluate d_s in terms of x_d, y_d in the detector plane.

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

$$= dz\sqrt{1 + (dx/dz)^2 + (dy/dz)^2}$$



Now finding expressions for x and y paths in terms of z.
We want to describe what is happening at some general location x,y in the body.

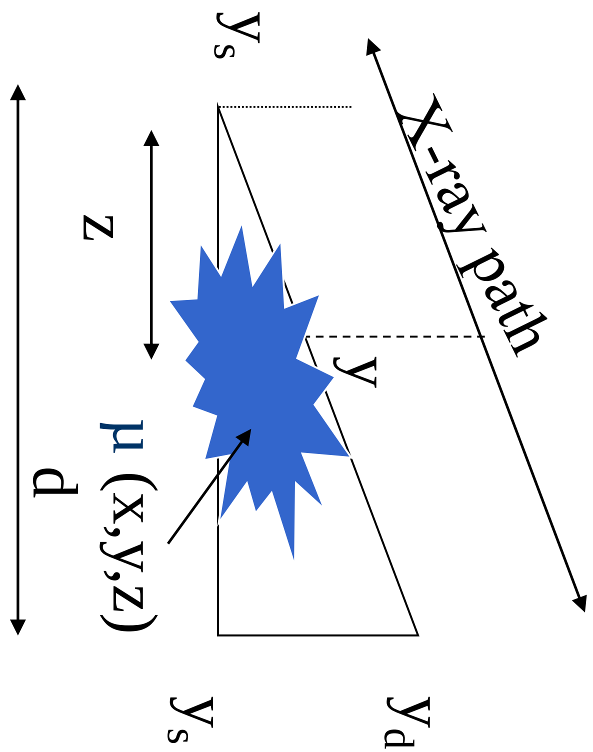
By similar triangles:

$$\frac{y_d - y_s}{y - y_s} = \frac{d}{z}$$

$$y_d - y_s \frac{z}{d} = \mathrm{y - y_s}$$

$$y = \frac{((y_d - y_s)z)}{d} + y_s$$

Similarly $$x = \frac{((x_d - x_s)z)}{d} + x_s$$



y-z plane shown

Again, we will build a parametric equation

$$\frac{dx}{dz} = \frac{(x_d - x_s)}{d} \qquad \frac{dy}{dz} = \frac{(y_d - y_s)}{d}$$

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = dz\sqrt{1 + (dx/dz)^2 + (dy/dz)^2}$$

$$ds = dz\sqrt{1 + (\frac{x_d - x_s}{d})^2 + (\frac{y_d - y_s}{d})^2} = dz\sqrt{1 + \frac{r_d^2}{d^2}}$$

$$dI_d = dI_i \exp[-\sqrt{1 + \frac{r_d^2}{d^2}} \int \mu_m (\frac{x_d - x_s}{d} z + x_s, \frac{y_d - y_s}{d} z + y_s, z)]dz$$

Recalling that $M = \frac{d}{z}\ and\ m = \frac{-(d-z)}{z}$

We can have the result at an arbitrary detector point and arbitrary source point.

$$dI_d = dI_i \exp[-\sqrt{1+\frac{r_d^2}{d^2}}\int \mu_m(\frac{x_d - mx_s}{M}, \frac{y_d - my_s}{M}, z)]dz$$

To get the entire result I_d (x_d, y_d), we add up the response from all the source points by integrating over the source.

$$I_d(x_d, y_d) = \int\int dI_d(x_d, y_d, x_s, y_s) =$$

$$\frac{1}{4\pi d^2}\iint \frac{s(x_s, y_s)}{(1+r_d^2/d^2)^{3/2}} \exp[-\sqrt{1+\frac{r_d^2}{d^2}}\int \mu_m(\frac{x_d - mx_s}{M}, \frac{y_d - my_s}{M}, z)dz]dx_s dy_s$$

Let's make some assumptions and simplify

1) Ignore both obliquity factors
2) Assume thin planar object at $z=z_o$ $\mu = \tau(x,y)\,\delta(z - z_o)$

$$I_d(x_d, y_d) = \frac{1}{4\pi d^2} \iint s(x_s, y_s) \exp[-\tau(\frac{x_d - mx_s}{M}, \frac{y_d - my_s}{M})] dx_s dy_s$$

To define this expression in a space invariant convolution form, let:

$$x_s' = mx_s \quad \text{and} \quad y_s' = my_s$$

And by considering a magnified source and Magnified object, we can get space invariance as:

$$I_d = \frac{1}{4\pi d^2 m^2} \iint s(\frac{x_s'}{m}, \frac{y_s'}{m}) \exp[-\tau(\frac{x_d - x_s'}{M}, \frac{y_d - y_s'}{M}) dx_s' dy_s'$$

or:

$$I_d = \frac{1}{4\pi d^2 m^2} s(\frac{x_d}{m}, \frac{y_d}{m}) ** \exp[-\tau(\frac{x_d}{M}, \frac{y_d}{M})$$

Let's make above expression linear by replacing:

$$t(x,y) = \exp[-\tau(x,y)]$$

Then:

$$I_d = \frac{1}{4\pi d^2 m^2} s(\frac{x_d}{m}, \frac{y_d}{m}) ** t(\frac{x_d}{M}, \frac{y_d}{M})$$

Here the collection efficiency of pinhole is divided by the ratio of image and source area i.e. m_2

Consider 10mm object 1mm source

z	$\|m\|$	M	Image size	Blur
d	0	1	10mm	0
d/2	1	2	20mm	1mm
d/10	9	10	100mm	9mm

Volumetric Object

What about volumetric objects? Ignoring obliquity, for thin plane we have:

$$I_d = \frac{1}{4\pi d^2 m^2} \iint s(x_s, y_s) \exp[-\tau(\frac{x_d - m x_s}{M}, \frac{y_d - m y_s}{M}) dx_s dy_s$$

Let's model object as an array of planes (||||||) τ (x,y)

Therefore: $\mu = \sum_i \tau_i (x,y)\, \delta (z - z_i)$

where m = m(z) or $m_i = -(d - z_i)/ z_i$ and $M_i = d/z_i$

$$\exp[-\tau(\frac{x_d - mx_s}{M}, \frac{y_d - my_s}{M})$$ term is still not linear however

Let’s assume the solid object as: $\int \mu \, dz = \sum \tau_i << 1$

Then we can linearize the exponent $\exp(-\sum \tau_i) = 1 - \sum \tau_I$

$$I_d = \frac{1}{4\pi d^2 m^2} \iint s(x_s, y_s)[1 - \sum \tau_i (\frac{x_d - m_i x_s}{M_i}, \frac{y_d - m_i y_s}{M_i})]$$

$I_d \approx I_i - \sum_i 1/(4\pi d^2 m_i^2)\, s(x_d/m_i, y_d/m_i) ** \tau_i (x_d/M_i, y_d/M_i)$

where $I_i = 1/(4\pi d^2) \int\int s(x_s,y_s)\, dx_s\, dy_s$

The output is seen as incident radiation minus a summation of convolutions. Good math, but poor approximation. This is true only in very thin regions of the body

Reasonable approximation (except at boundaries μ approaches 0)

Where μ of water is considered as mean μ, and departure from that is μ_Δ

$$\mu = \mu_{mean} + \mu_\Delta$$

$$\exp [-\int(\mu_m + \mu_\Delta)\, dz = \exp [- \int\mu_m\, dz] \exp [- \int\mu_\Delta\, dz]$$

Since $\int\mu_\Delta dz << 1$ We can linearize with the approximation

$$\approx [\exp -\int\mu_m\, dz] [1 - \int\mu_\Delta\, dz]$$

$$I_d = \frac{1}{4\pi d^2 m^2} \iint s(x_s, y_s) \exp[-\int \mu_m (\frac{x_d - m x_s}{M}, \frac{y_d - m y_s}{M}, z)] dz \times$$

$$[1 - \int \mu_\Lambda (\frac{x_d - m x_s}{M}, \frac{y_d - m y_s}{M}, z)] dx_s dy_s$$

Since μ_m is a constant, the first exponential term is a constant.

Since μ_m is a constant, we can make the approximation:

$$\mu_m(\frac{x_d - mx_s}{M}, \frac{y_d - my_s}{M}, z) \cong \mu_m(\frac{x_d}{M}, \frac{y_d}{M}, z)$$

The remaining integration can now be seen as a series of thin planes.

$$I_d = [\exp - \int \mu_m\ (x_d/M, y_d/M, z)\ dz\ ] \bullet$$
$$[\ I_i - \sum 1/(4\pi d^2 m_i^2)\ s\ (x_d/m_i, y_d/m_i)\ {**}\ \tau_{\Delta i}\ (x_d/M_i, y_d/M_i)$$

$$I_d = T_{water}\ (x_d, y_d)\ [\ I_i - \sum 1/(4\pi d^2 m_i^2)$$
$$\bullet\ s\ ((x_d/m_i), (y_d/m_i))\ {**}\ \tau_{\Delta i}\ (x_d/M_i, y_d/M_i)]$$

where $T_{water} = [\exp - \int \mu_m\ (x_d/M, y_d/M, z)\ dz\ ]$
and $I_i = 1/(4\pi d^2) \int\int s(x_s,y_s)\ dx_s\ dy_s$

Thus, we have justified viewing the body as an array of planes.