Lecture:
Linear systems and convolution

Linearity Conditions

Let $f_1(x,y)$ and $f_2(x,y)$ describe two objects we want to image.

$f_1(x,y)$ can be any object and represent any characteristic of the object.
(e.g. color, intensity, temperature, texture, X-ray absorption, etc.)

Assume each is imaged by some imaging device (system).

Let $f_1(x,y) \rightarrow g_1(x,y)$

$f_2(x,y) \rightarrow g_2(x,y)$

Let's scale each object and combine them to form a new object.

$$a\, f_1(x,y) + b\, f_2(x,y)$$

If the system is linear, output is

$$a\, g_1(x,y) + b\, g_2(x,y)$$

Linearity Example:

$$x \rightarrow \boxed{} \rightarrow \sqrt{x}$$

Is this a linear system?

Linearity Example:

$$x \rightarrow \boxed{} \rightarrow \sqrt{x}$$

Is this a linear system?

$$9 \rightarrow 3$$

$$\underline{16} \rightarrow \underline{4}$$

$$9 + 16 = 25 \rightarrow 5 \qquad 3 + 4 \neq 5$$

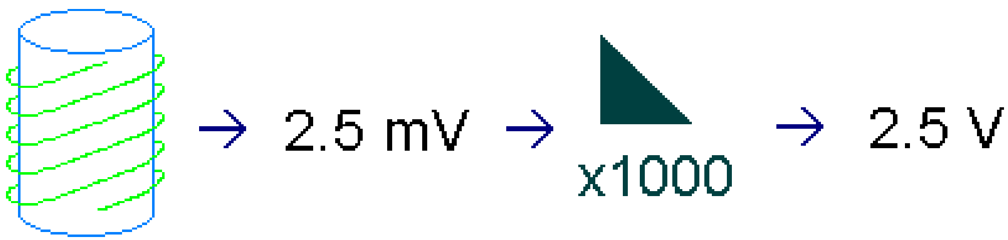
Not linear.

Example in medical imaging:

Doubling the X-ray photons → doubles those transmitted

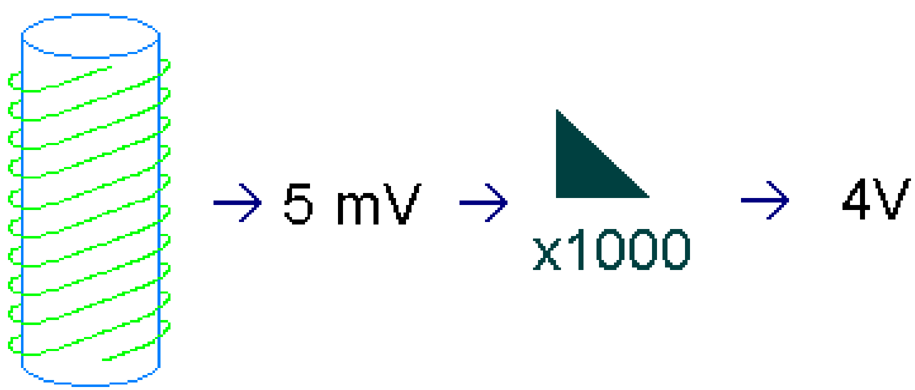
Doubling the nuclear medicine source energy → doubles the reception

MR:



Maximum output voltage is 4 V.

Now double the water:



The A/D converter system is non-linear after 4 mV. (overranging)

Linearity allows decomposition of functions

Linearity allows us to decompose our input into smaller, elementary objects.

Output is the sum of the system's response to these basic objects.

Elementary Function:

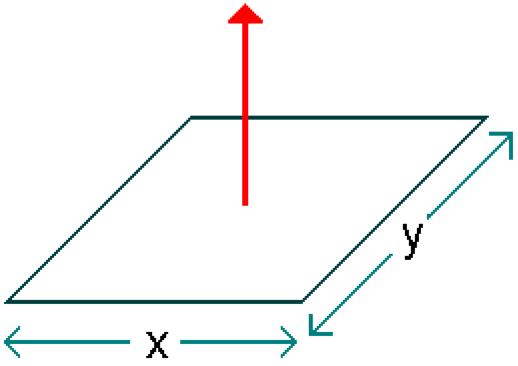
The two-dimensional delta function δ(x,y)

δ(x,y) has infinitesimal width and infinite amplitude.

Key: Volume under function is 1.

$$\int_{-\infty}^{\infty}\int \delta(x,y)\,dx\,dy = 1$$

δ(x,y)



The delta function as a limit of another function

Powerful to express δ(x,y) as the limit of a function

Gaussian: $$\lim_{a \to \infty} a^2 \exp[-\pi a^2(x^2+y^2)] = \delta(x,y)$$

2D Rect Function: $$\lim_{a \to \infty} a^2 \, \Pi(ax)\, \Pi(ay) = \delta(x,y)$$

where $\Pi(x) = 1$ for $|x| < ½$

For this reason, $\delta(bx) = (1/|b|)\, \delta(x)$

Sifting property of the delta function

$$f(x_1, y_1) = \int_{-\infty}^{\infty}\int f(\varepsilon,\eta)\delta(x_1-\varepsilon,\ \ y_1-\eta)d\varepsilon d\eta$$

The delta function at $x_1 = \varepsilon$, $y_1 = \eta$ has sifted out $f(x_1,y_1)$ at that point.

One can view $f(x_1,y_1)$ as a collection of delta functions,

each weighted by $f(\varepsilon,\eta)$.

Imaging Analyzed with System Operators

From previous page:

$$f(x_1, y_1) = \int\int_{-\infty}^{\infty} f(\varepsilon, \eta)\delta(x_1 - \varepsilon, y_1 - \eta)d\varepsilon d\eta$$

Let system operator (imaging modality) be ζ, so that

$$g(x_2, y_2) = \zeta[f(x_1, y_1)]$$

Why the new coordinate system (x_2, y_2)*?*

Imaging Analyzed with System Operators

$$f(x_1, y_1) = \iint_{-\infty}^{\infty} f(\varepsilon, \eta)\delta(x_1 - \varepsilon, y_1 - \eta)\, d\varepsilon\, d\eta$$

$$g(x_2, y_2) = \zeta[f(x_1, y_1)]$$

Then,

$$\zeta\big[f(x_1, y_1)\big] = \zeta\Big[\iint_{-\infty}^{\infty} f(\varepsilon, \eta)\cdot\delta(x_1 - \varepsilon, y_1 - \eta)\, d\varepsilon\, d\eta\Big]$$

↑ Generally blurred object

↑ System operating on entire input object $f_1\,(x_1, y_1)$

By linearity, we can consider the output as a sum of the outputs from all the weighted elementary delta functions.

Then,

$$g(x_2, y_2) = \int_{-\infty}^{\infty}\int f(\varepsilon, \eta)\,\zeta\big[\delta(x_1 - \varepsilon, y_1 - \eta)\big]\, d\varepsilon\, d\eta$$

System response to a two-dimensional delta function

$$h(x_2, y_2; \varepsilon, \eta) = \zeta\left[\delta(x_1 - \varepsilon, y_1 - \eta)\right]$$

Substituting this into

$$g(x_2, y_2) = \int_{-\infty}^{\infty}\int f(\varepsilon, \eta)\,\zeta\left[\delta(x_1 - \varepsilon, y_1 - \eta)\right] d\varepsilon\, d\eta$$

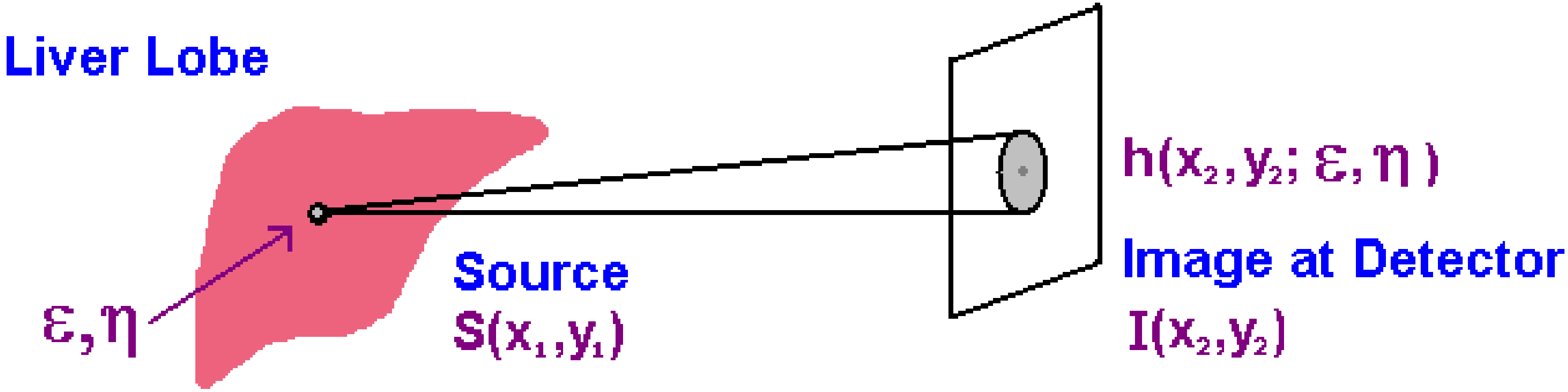
yields the Superposition Integral

$$g(x_2, y_2) = \iint_{-\infty}^{\infty} f(\varepsilon, \eta) \cdot h(x_2, y_2; \varepsilon, \eta)\, d\varepsilon\, d\eta$$

Example in medical imaging:

Consider a nuclear study of a liver with a tumor point source

at $x_1 = \varepsilon$, $y_1 = \eta$



Radiation is detected at the detector plane.

To obtain a general result, we need to know all combinations

$$h(x_2, y_2; \varepsilon, \eta)$$

By "general result", we mean that we could calculate

the image $I(x_2, y_2)$ for any source input $S(x_1, y_1)$

Time invariance

A system is time invariant if its output depends only on relative time of the input, not absolute time. To test if this quality exists for a system, delay the input by t_0. If the output shifts by the same amount, the system is time invariant

i.e.

$$f(t) \rightarrow g(t)$$

$$f(t - t_o) \rightarrow g(t - t_o)$$

input delay output delay

Is $f(t) \rightarrow \boxed{f(at)} \rightarrow g(t)$ (an audio compressor) time invariant?

Time invariance

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input delay   output delay

Is $f(t) \rightarrow \boxed{f(at)} \rightarrow g(t)$ (an audio compressor) time invariant?

$f(t - t_o) \rightarrow \boxed{f(at)} \rightarrow f(a(t - t_o))$ - output of audio compressor

$\neq f(at - t_o)$ - shifted version of output (this would be a time invariant system.)

So $f(t) \rightarrow f(at) = g(t)$ is not time invariant.

Space or shift invariance

A system is space (or shift) invariant if its output depends only on relative position of the input, not absolute position.

If you shift input in the plane, $\longrightarrow$ The response shifts, but the shape of the response stays the same.

If the system is shift invariant,

$$h(x_2, y_2; \varepsilon, \eta) = h(x_2-\varepsilon, y_2-\eta)$$

and the superposition integral becomes the 2D convolution function:

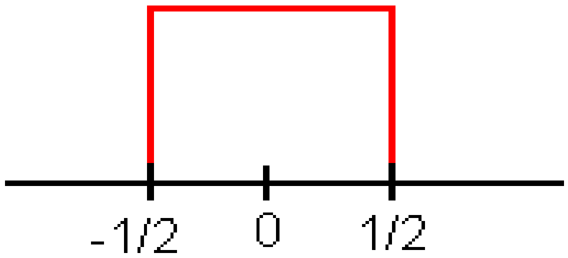
$$g(x_2, y_2) = \iint_{-\infty}^{\infty} f(\varepsilon,\eta) \cdot h(x_2-\varepsilon, y_2-\eta)\, d\varepsilon\, d\eta$$

Notation: $g = f{*}{*}h$ (** sometimes implies two-dimensional convolution, as opposed to $g = f*h$ for one dimension. Often we will use $*$ with 2D and 3D functions and imply 2D or 3D convolution.)

One-dimensional convolution example:

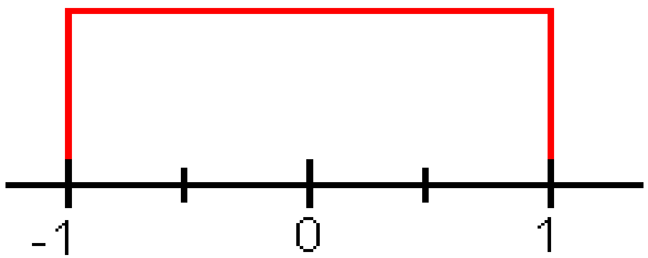
$g(x) = \Pi(x) * \Pi(x/2)$

Recall: $\Pi(x) = 1$ for $|x| < ½$



$\Pi(x/2) = 1$ for $|x|/2 < ½$

or $|x| < 1$



$$g(x) = \int_{-\infty}^{\infty} \Pi\left(\frac{x'}{2}\right) \cdot \Pi(x - x')\,dx'$$

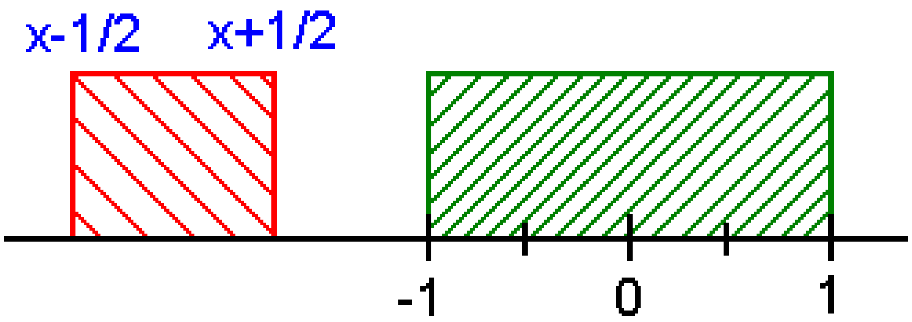
Flip one object and drag across the other.

flip delay

$$g(x) = \int_{-\infty}^{\infty} \Pi\left(\frac{x'}{2}\right) \cdot \Pi(-(x' - x))\,dx'$$

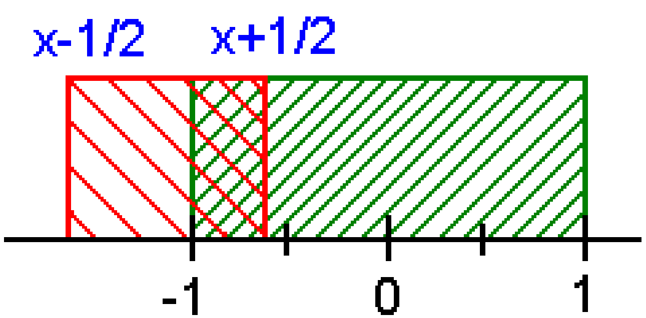
One-dimensional convolution example, continued:

Case 1: no overlap of Π(x-x’) and Π(x’/2)



$$g(x) = 0 \qquad \text{for} \quad x < -\frac{3}{2}$$

Case 2: partial overlap of Π(x-x’) and Π(x’/2)



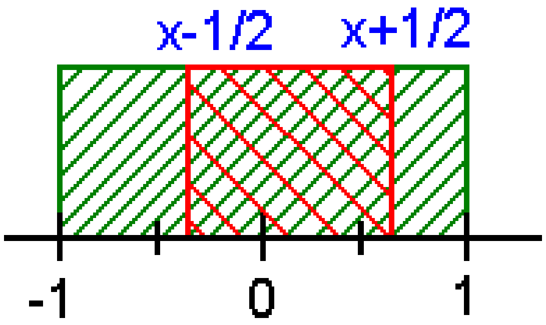
$$g(x) = \int_{-1}^{x+1/2} 1 \cdot 1 \cdot dx'$$

$$= x\Big]_{-1}^{x+1/2} = x + \frac{1}{2} - -1$$

$$g(x) = x + \frac{3}{2} \qquad \text{for} \; -\frac{3}{2} < x < -\frac{1}{2}$$

One-dimensional convolution example, continued(2):

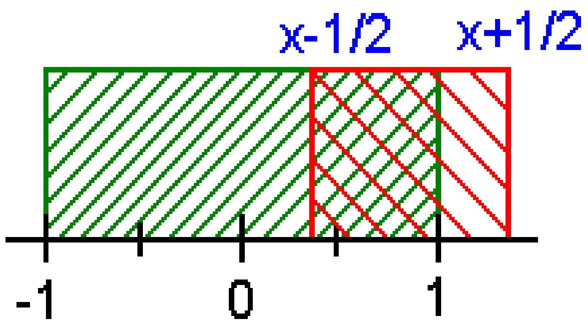
Case 3: complete overlap



$$g(x) = \int_{x-1/2}^{x+1/2} 1 \cdot 1 \cdot dx'$$

$$g(x) = 1 \qquad \text{for } -\tfrac{1}{2} < x < \tfrac{1}{2}$$

Case 4: partial overlap

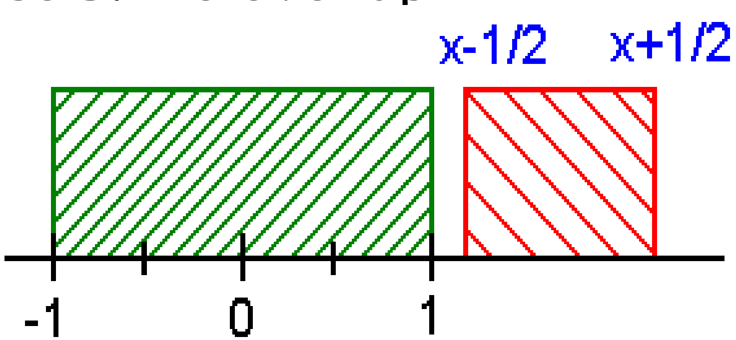


$$g(x) = \int_{x-1/2}^{1} 1 \cdot 1 \cdot dx'$$

$$= x\Big]_{x-1/2}^{1} = 1 - (x - \tfrac{1}{2})$$

$$g(x) = -x + \tfrac{3}{2} \qquad \text{for } \tfrac{1}{2} < x < \tfrac{3}{2}$$

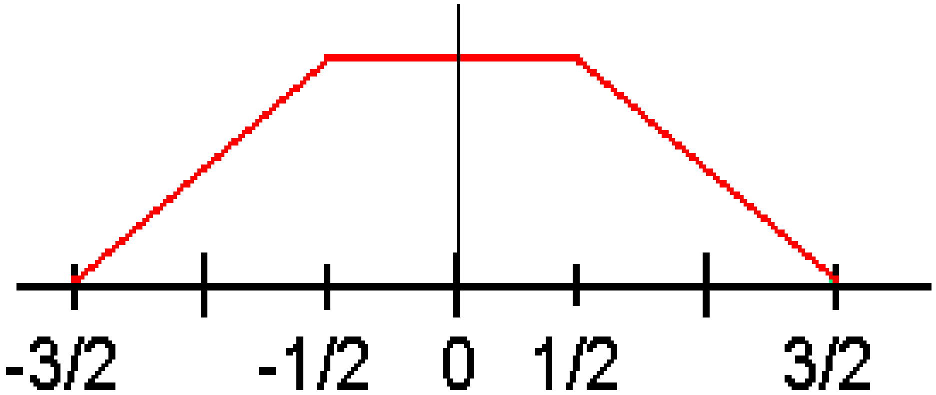
Case 5: no overlap



$$g(x) = 0 \qquad \text{for } x > \tfrac{3}{2}$$

One-dimensional convolution example, continued(3):

Result of convolution:



$$g(x) = 0 \qquad \text{for} \qquad x < -\frac{3}{2}$$

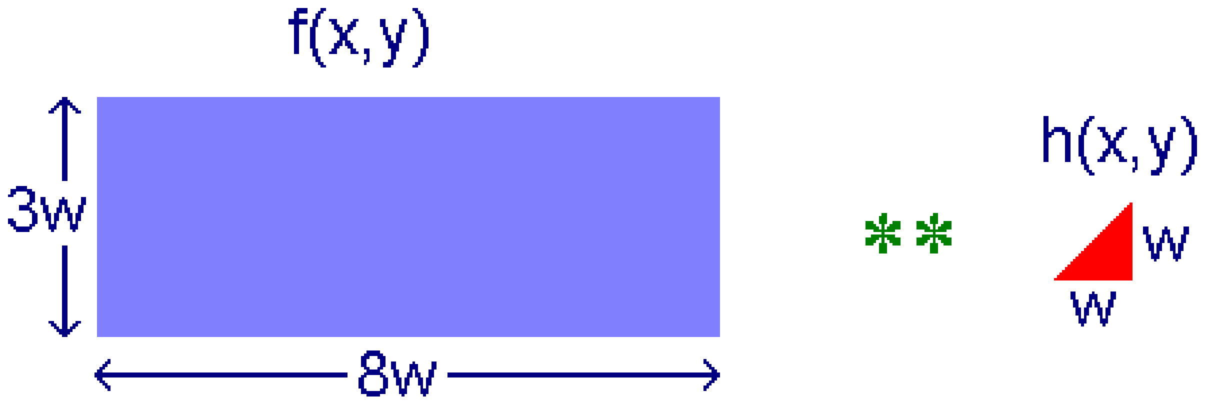
$$g(x) = x + \frac{3}{2} \qquad \text{for} \quad -\frac{3}{2} < x < -\frac{1}{2}$$

$$g(x) = 1 \qquad \text{for} \quad -\frac{1}{2} < x < \frac{1}{2}$$

$$g(x) = -x + \frac{3}{2} \qquad \text{for} \quad \frac{1}{2} < x < \frac{3}{2}$$

$$g(x) = 0 \qquad \text{for} \qquad x > \frac{3}{2}$$

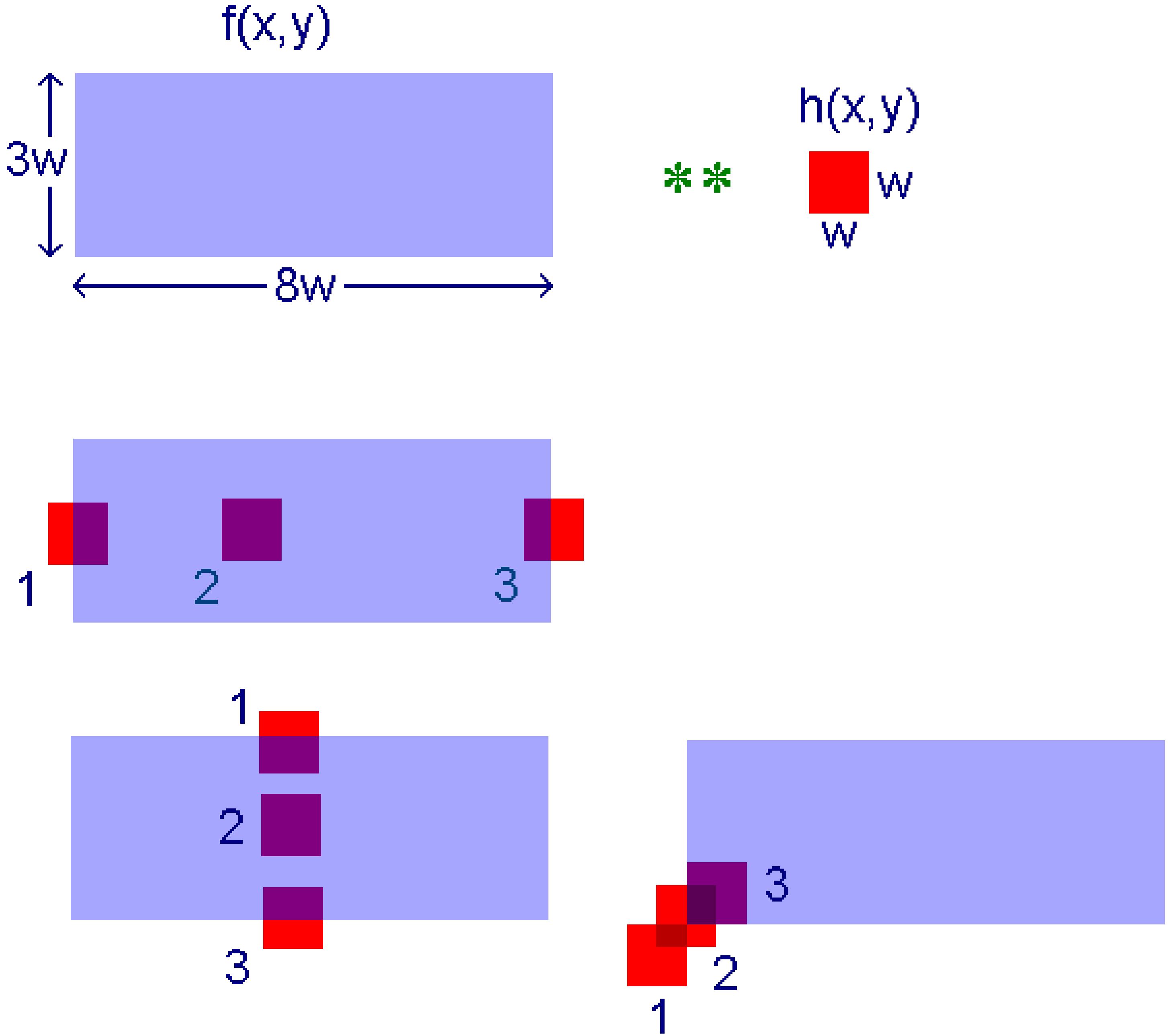
Two-dimensional convolution



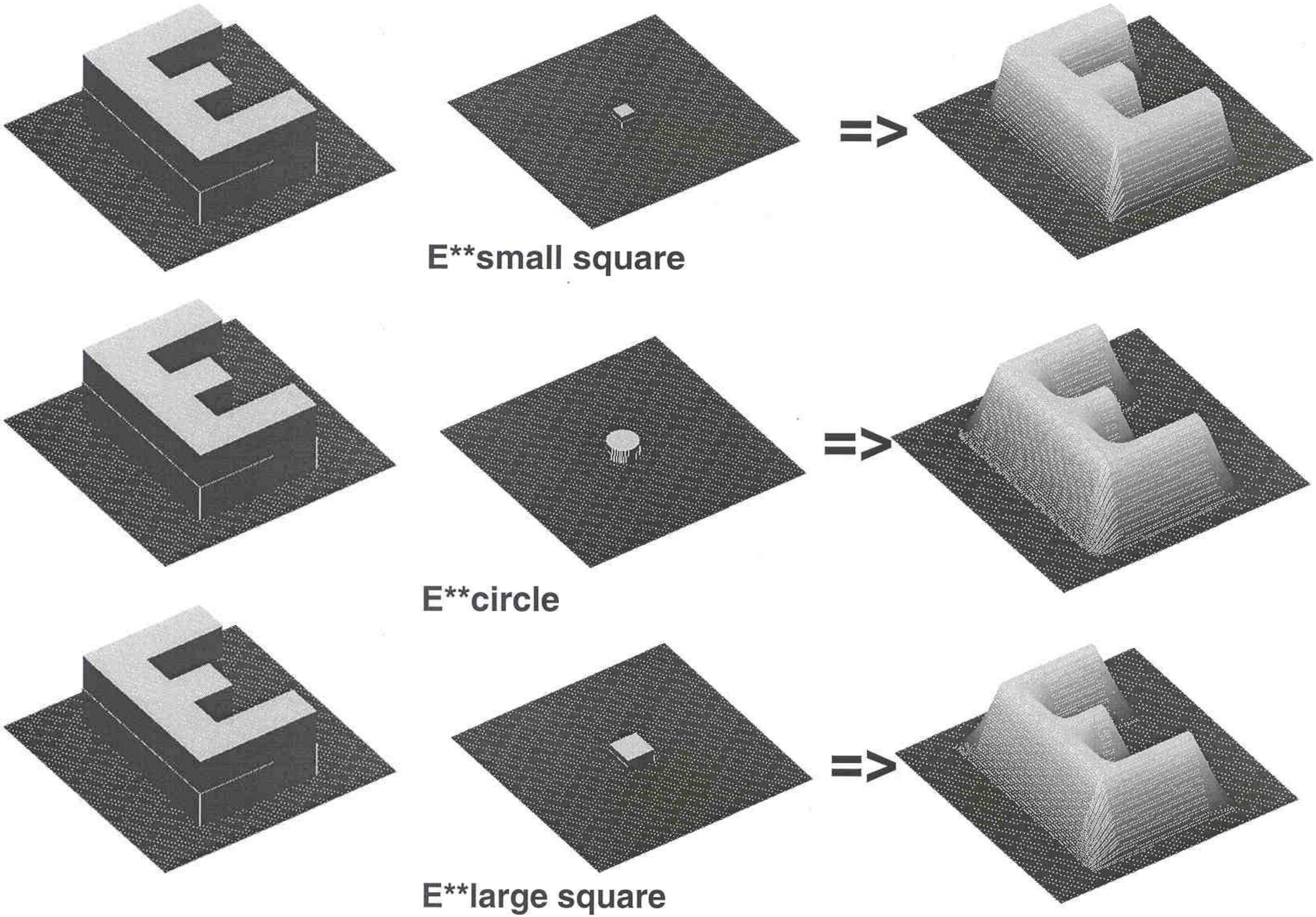
$$g(x,y) = \iint_{-\infty}^{\infty} f(x',y') \cdot h(x-x',y-y')\,dx'\,dy'$$

← sliding of flipped object

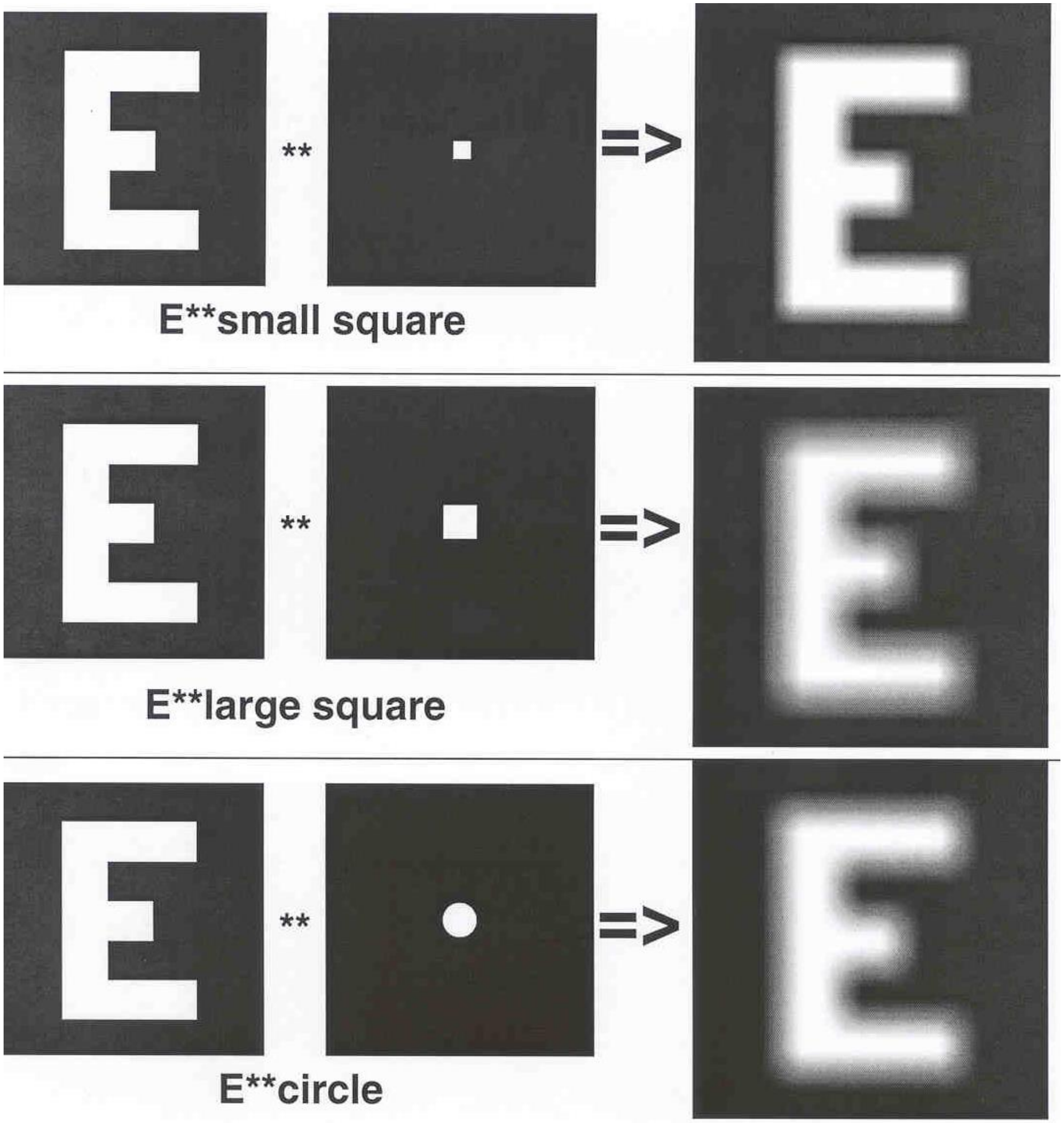
2D Convolution of a square with a rectangle.



2D Convolution of letter E - 3D plots



2D Convolution of letter E - Grayscale images



Review of Fourier Transforms

تبديل فوريه (Fourier Transform)

- پس ازعبور نور از يك منشور (Prism) يا diffraction grating، نور به اجزا مختلف با فركانس هاي خاص خود (مونوكروماتيك) تجزيه مي شود.
- اين امر مشابه تبديل فوريه (FT) است.
- مي توان يـك سيگنال يك بعدي را بصورت مجموعه اي از امواج سينوسي (با فركانس و دامنه متفاوت) نشان داد.
- هرچه فركانس هاي بيشتري را محاسبه نماييم تخمين فوريه يك سيگنال دقيق تر مي شود و اطلاعات بيشتري درباره شكل اوليه آن بدست مي آيد.

تبديل فوريه (Fourier Transform)

- FT مبتني بر اين واقعيت است كه سيگنال دوره اي (Periodic) شامل بي نهايت سيگنال هاي سينوسي وزن دار با فـركانس هاي متفاوت است. اين فركانس ها عبارتتند از فركانس پايه (Fundamental frequency) و مضارب درست اين فركانس پايه.
- **در تبديل فوريه، توابع پايه‌اي هم جهت (orthonormal basis function)، امواج سينوسي با فركانس‌هاي متفاوت هستند كه در فضاي بي‌نهايت تعريف شده‌اند**

تبديل فوريه (Fourier Transform)

- هر يك از ضرايب حاصل در تبديل فوريه توسط ضرب نقطه‌اي (inner product) تابع ورودي و يكي از توابع پايه‌اي (basis function) بدست مي‌آيد.
- اين ضرايب، در واقع، درجه شباهت بين تابع ورودي و تابع پايه‌اي مورد نظر را نشان مي‌دهد.
- اگر دو تابع پايه‌اي بر هم عمود (orthogonal) باشند، حاصل‌ضرب نقطه‌اي آنها صفر و لذا نشان مي‌دهد كه آن‌دو با هم شبيه نيستند.
- بنابراين اگر سيگنال يا تصوير ورودي از اجزايي تشكيل شده باشد كه يك يا چند تابع پايه‌اي داشته باشد، سپس آن يك يا چند ضريب بزرگ و ديگر ضرايب كوچك هستند.

Inverse Fouriers Transform

- در تبديل معكوس، سيگنال يا تصوير اوليه توسط مجموع توابع پايه‌اي (در فركانس‌هاي مختلف) كه تحت تاثير وزن ضرايب تبديل قرار گرفته‌اند، بازسازي مي‌شود.

- بنابراين اگر يك سيگنال يا تصوير از اجزائي شبيه به تعداد معدودي از توابع پايه‌اي تشكيل شده باشد، بسياري از عبارات موجود در اين جمع (ضرايب تبديل) حذف شده و فقط تعدادي از اين ضرايب تبديل، تقويت اجزايي از تصوير را كه شبيه به توابع مربوطه پايه‌اي است انجام داده و تصوير را مي‌سازند.

Advantage

- وقتي تبديل فوريه يك سيگنال يا تصوير بدست مي آيد، اعمال متعدد رياضي برروي آنها قابل انجام است. در فضاي فركانسي انجام اين عمليات رياضي از انجام آنها در فضاي مكاني به مراتب ساده تر است.

- بعنوان مثال عمل Convolution به يك ضرب ساده تبديل مــي شود و روش هاي پردازشي ديگر نيز مانند Correlation، differentiation، integration و Interpolation به سهولت انجام مي شوند.

1-D Fourier Theory

$$f(t) \leftrightarrow F(f)$$

Fourier Transform:

$$F(f) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i \cdot 2\pi \cdot f \cdot t} dt$$

Inverse Fourier Transform:

$$f(t) = \int_{-\infty}^{\infty} F(f) \cdot e^{+i \cdot 2\pi \cdot f \cdot t} df$$

You may have seen the Fourier transform and its inverse written as:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i \cdot \omega \cdot t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{+i \cdot \omega \cdot t} d\omega$$

Why use the top version instead?

1) No scaling factor ($1/2\pi$); easier to remember.
2) Easier to think in Hz than in radians/s

Review of 1-D Fourier Theory, continued

Let's generalize so we can consider functions of variables other than time.

$$\mathrm{f}(x) \leftrightarrow \mathrm{F}(u)$$

Fourier Transform:

$$\mathrm{F}(u) = \int_{-\infty}^{\infty} \mathrm{f}(x) \cdot \mathrm{e}^{-i \cdot 2\pi \cdot u \cdot x} dx$$

Inverse Fourier Transform:

$$\mathrm{f}(x) = \int_{-\infty}^{\infty} \mathrm{F}(u) \cdot \mathrm{e}^{+i \cdot 2\pi \cdot x \cdot u} du$$

Review of 1-D Fourier theory, continued (2)

$$F(u) = \int_{-\infty}^{\infty} f(x) \cdot e^{-i \cdot 2\pi \cdot u \cdot x} dx \qquad f(x) = \int_{-\infty}^{\infty} F(u) \cdot e^{+i \cdot 2\pi \cdot x \cdot u} du$$

$$e^{-i \cdot 2\pi \cdot u \cdot x} = \cos(2\pi \cdot u \cdot x) - i \cdot \sin(2\pi \cdot u \cdot x)$$

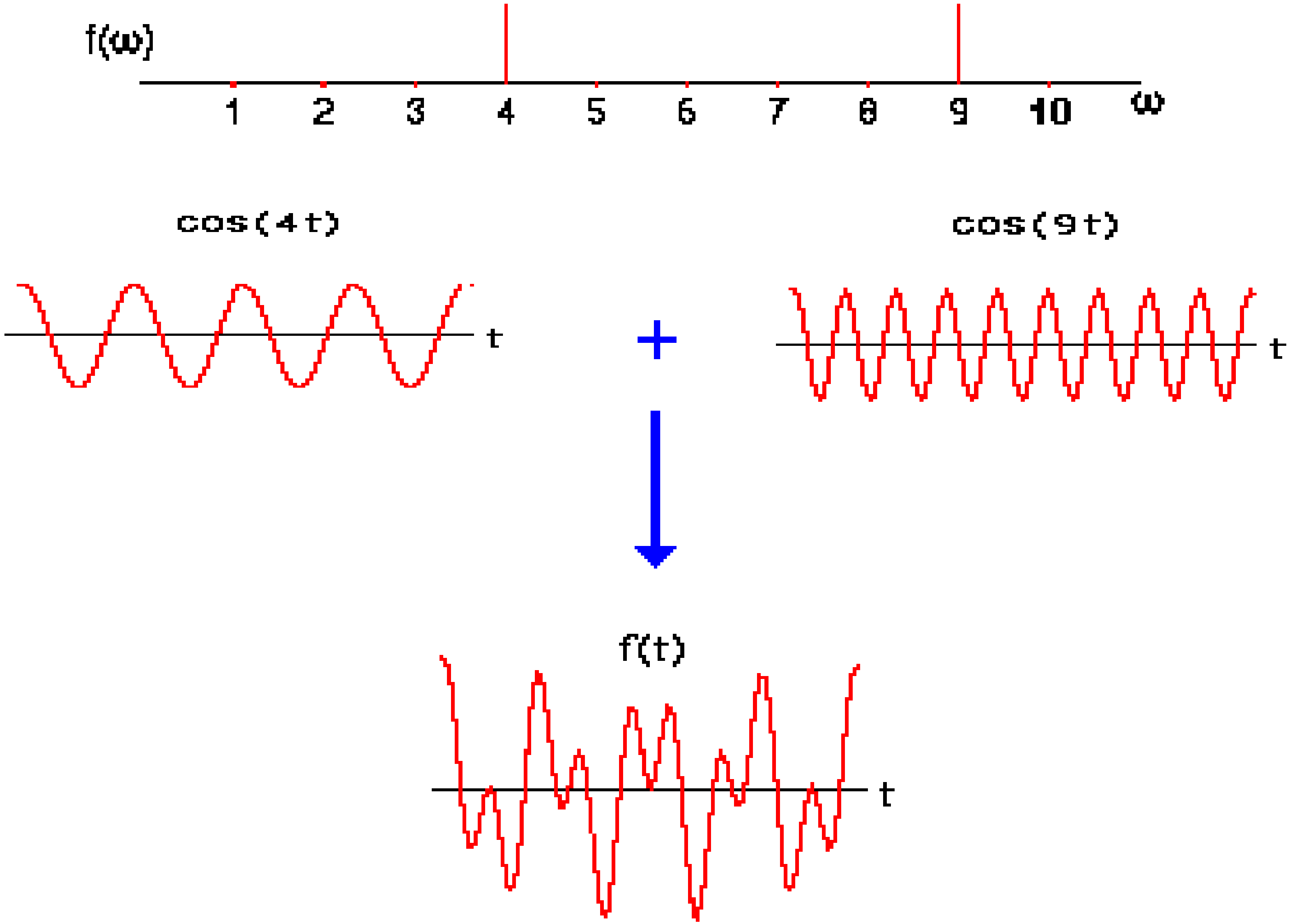
↑ Orthogonal basis functions

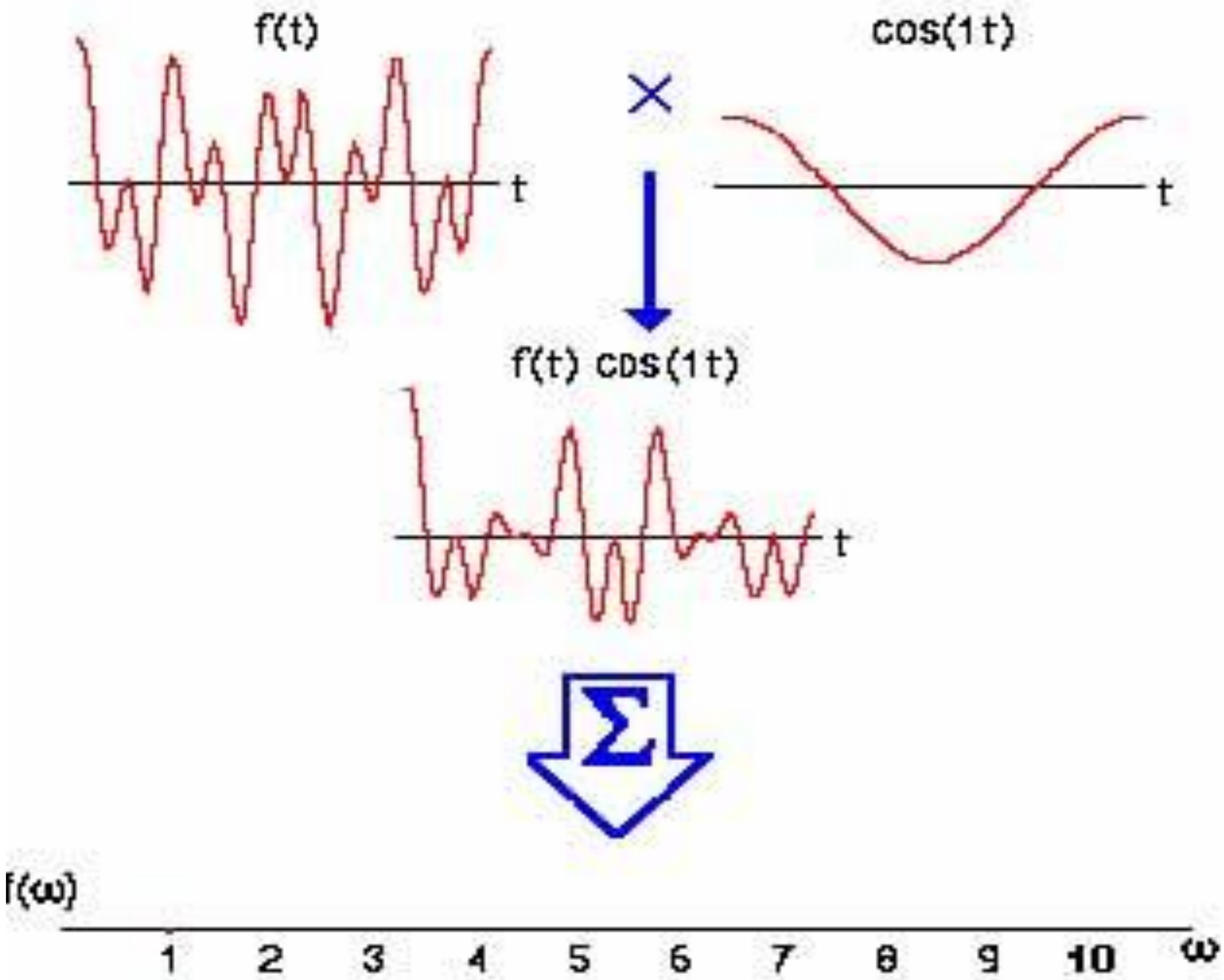
f(x) can be viewed as as a linear combination of the complex exponential basis functions.

F(u) gives us the magnitude and phase of each of the exponentials that comprise f(x).

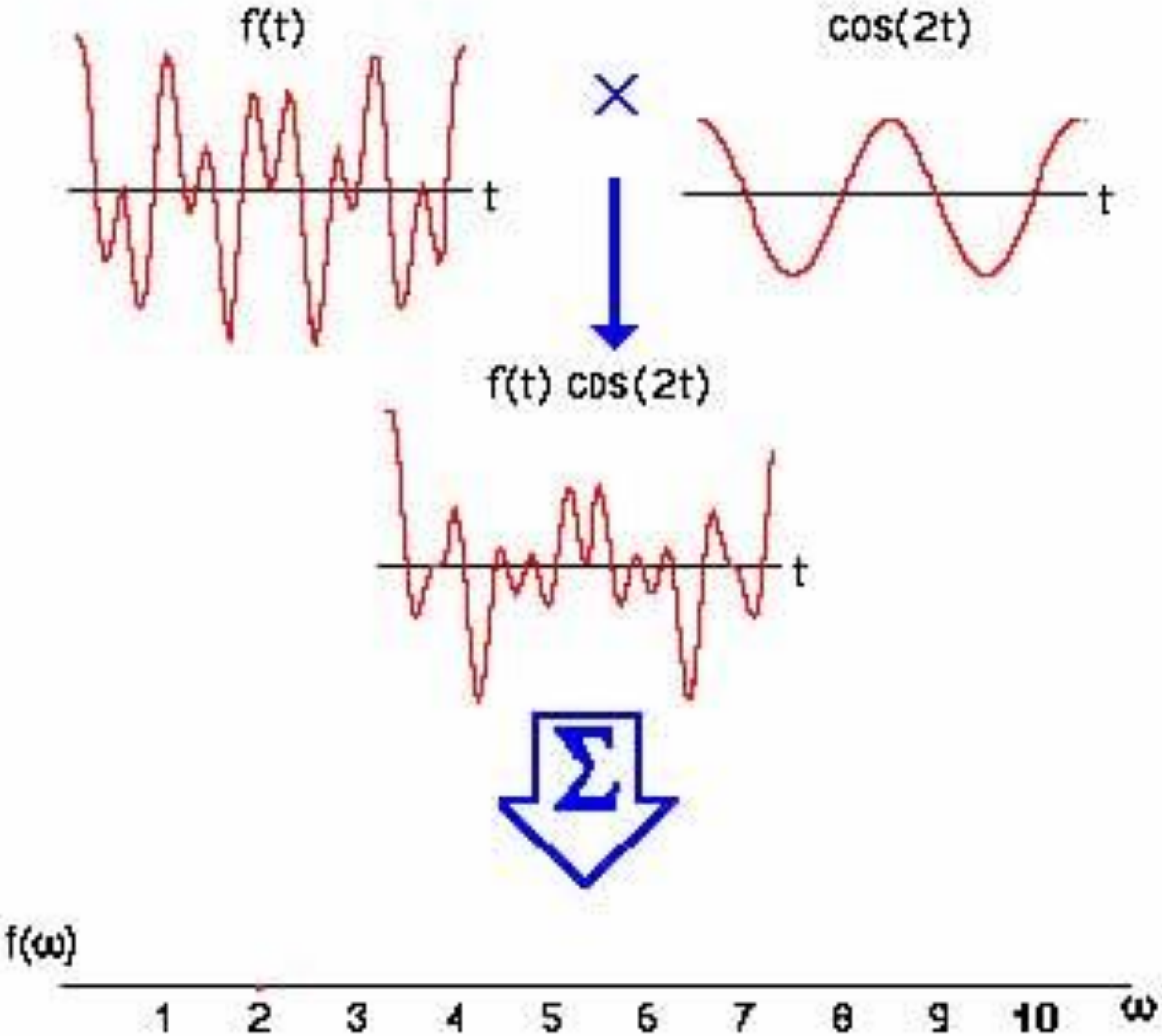
In fact, the Fourier integral works by sifting out the portion of f(x) that is comprised of the the function exp(+i 2π u_o x).

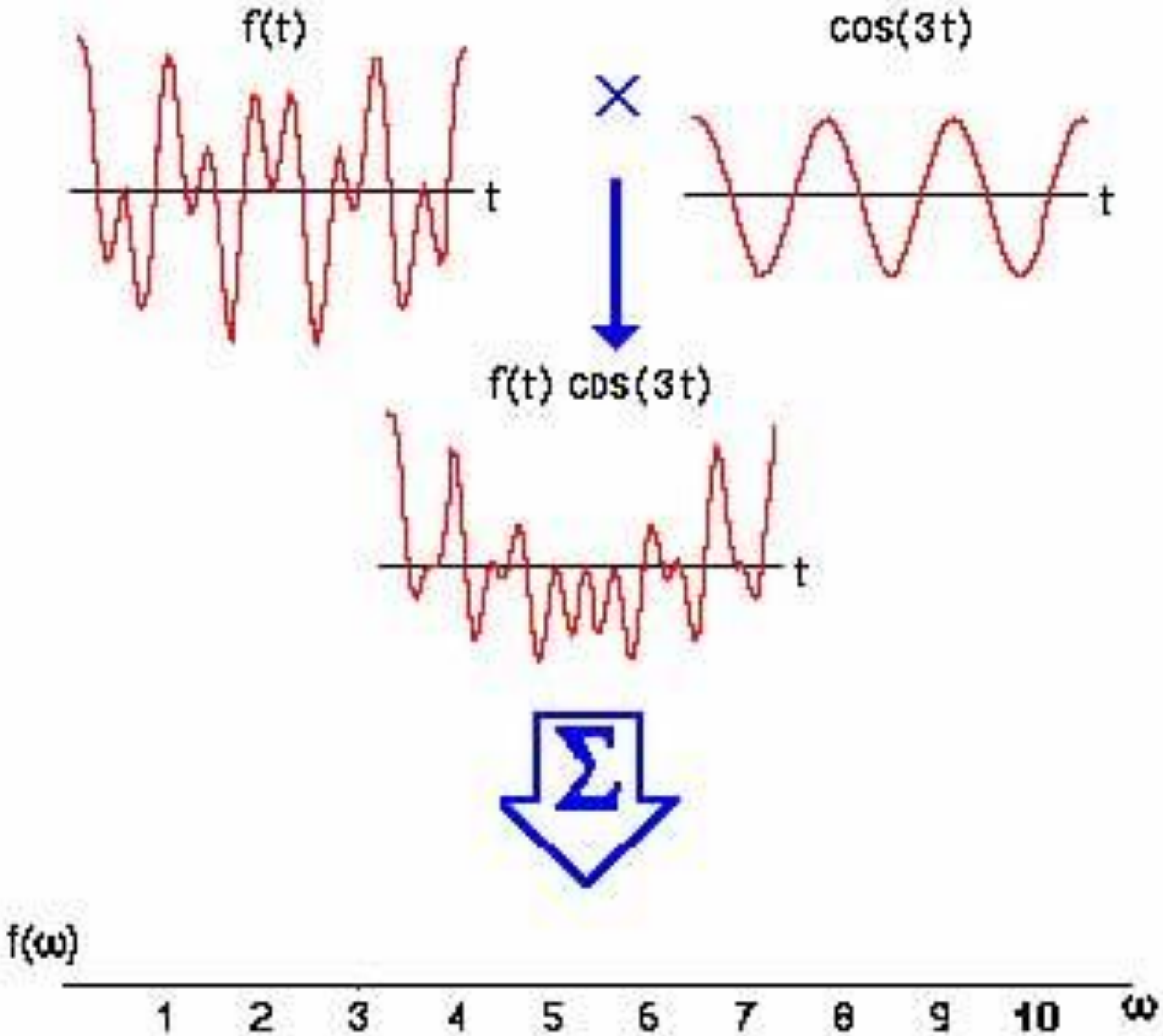
$$f(t) = \cos(4t) + \cos(9t)$$

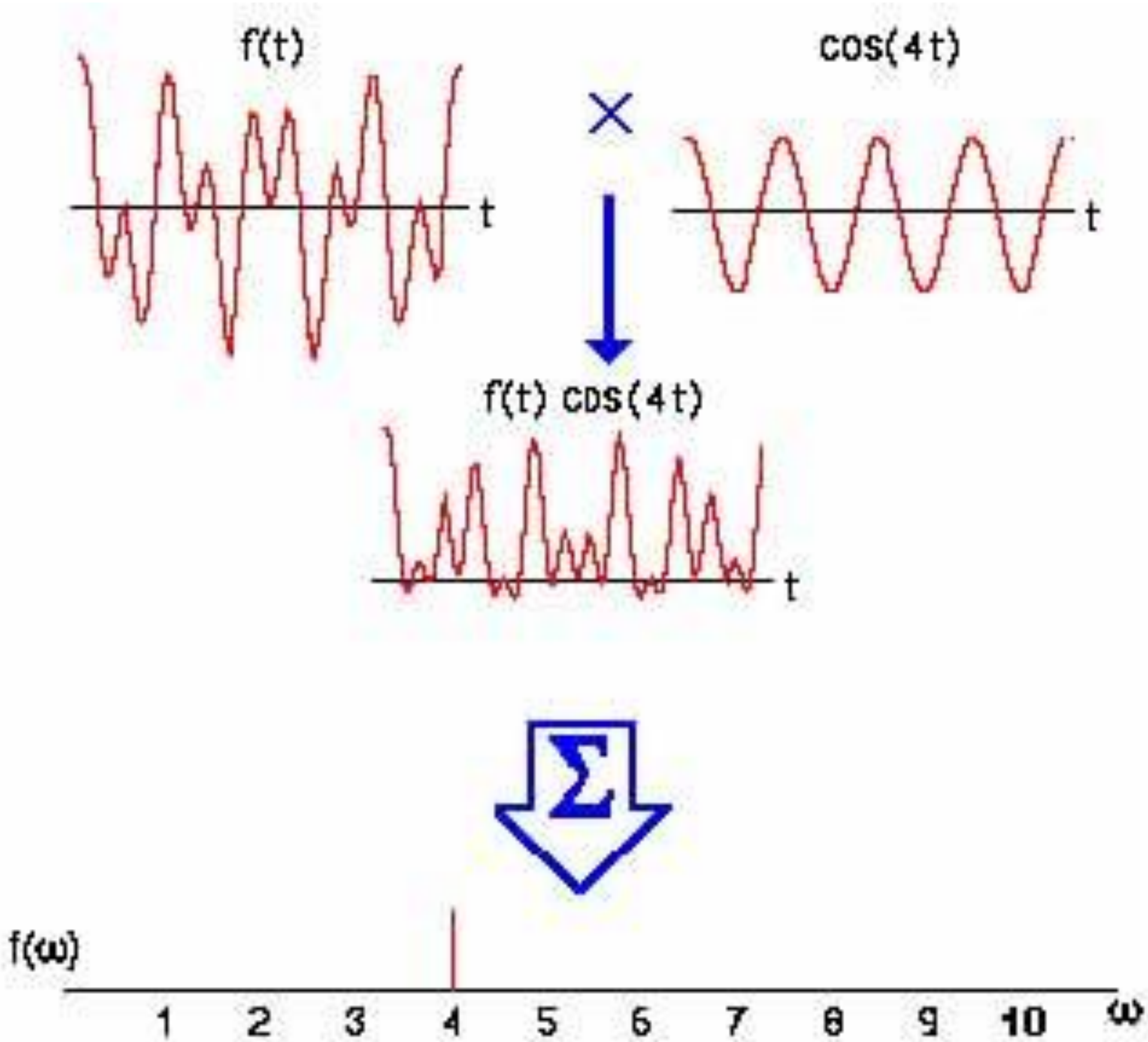


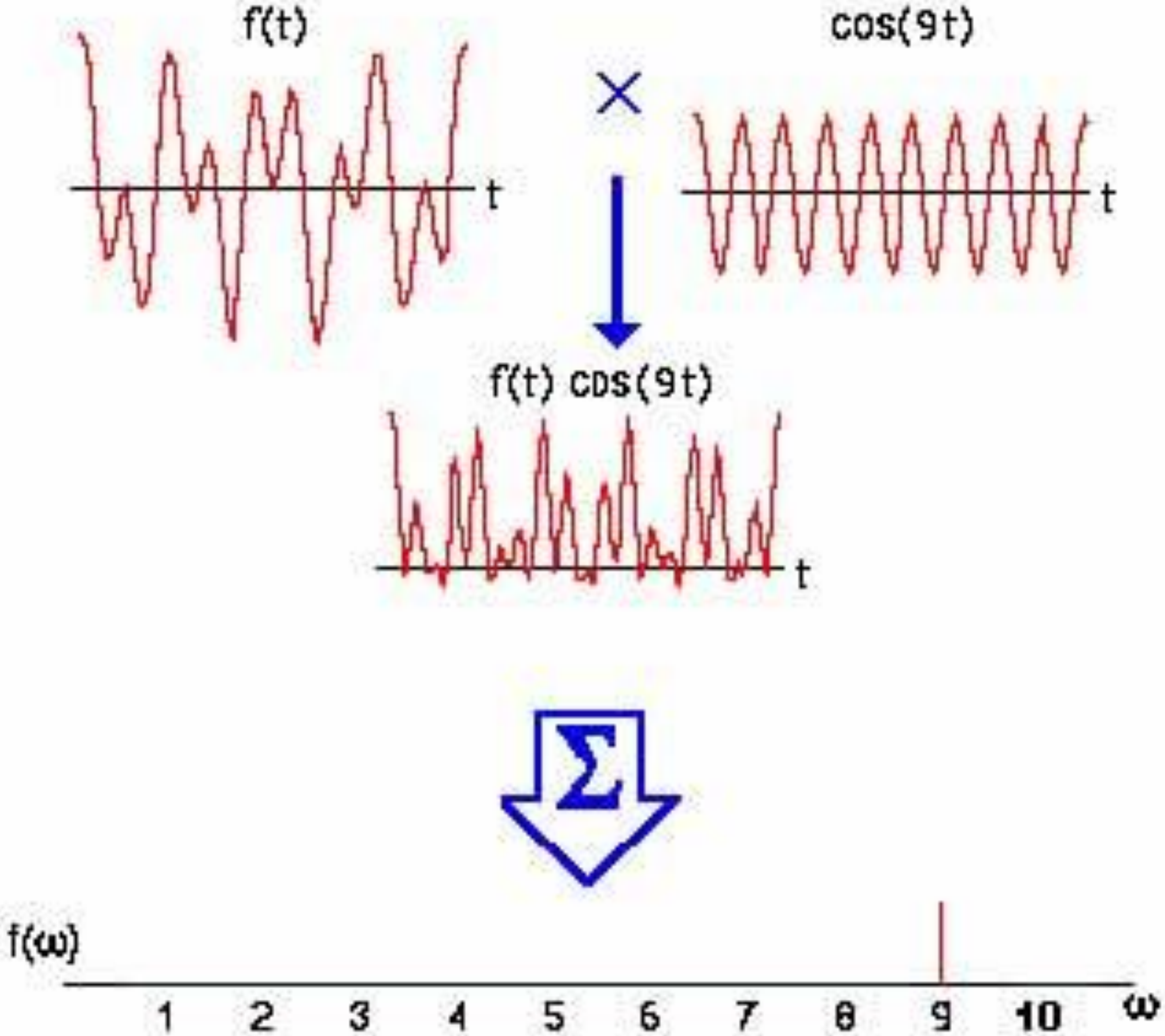


f(t)
cos(1t)
×
t
f(t) cos(1t)
Σ
f(ω)
1
2
3
4
5
6
7
8
9
10
ω


f(t)
cos(2t)
t
t
f(t) cos(2t)
t
Σ
f(ω)
1 2 3 4 5 6 7 8 9 10
ω


f(t)
cos(3t)
×
t
t
f(t) cos(3t)
t
Σ
f(ω)
1 2 3 4 5 6 7 8 9 10
ω


f(t)
cos(4t)
t
t
f(t) cos(4t)
t
Σ
f(ω)
1 2 3 4 5 6 7 8 9 10
ω



f(t)
cos(9t)
t
t
f(t) cos(9t)
t
Σ
f(ω)
1
2
3
4
5
6
7
8
9
10
ω

Some Fourier Transform Pairs and Definitions

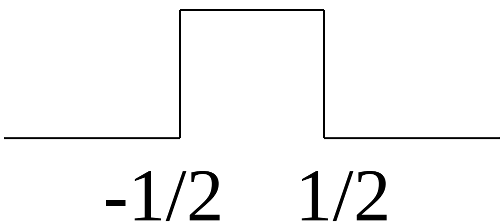
$$\mathrm{f}(x) \leftrightarrow \mathrm{F}(\mathrm{u})$$

$$\sin(2\pi x) \leftrightarrow \frac{1}{2i}\left(\delta(u-1)-\delta(u+1)\right)$$

$$\cos(2\pi x) \leftrightarrow \frac{1}{2}\left(\delta(u-1)+\delta(u+1)\right)$$

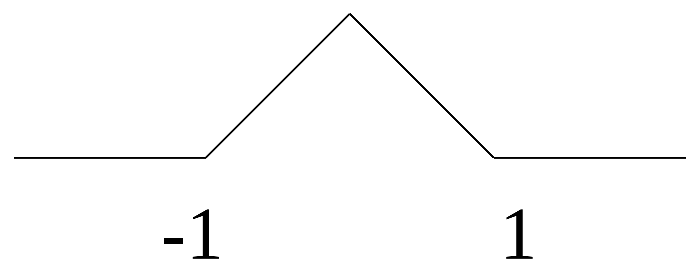
$$e^{-\pi x^2} \leftrightarrow e^{-\pi u^2}$$

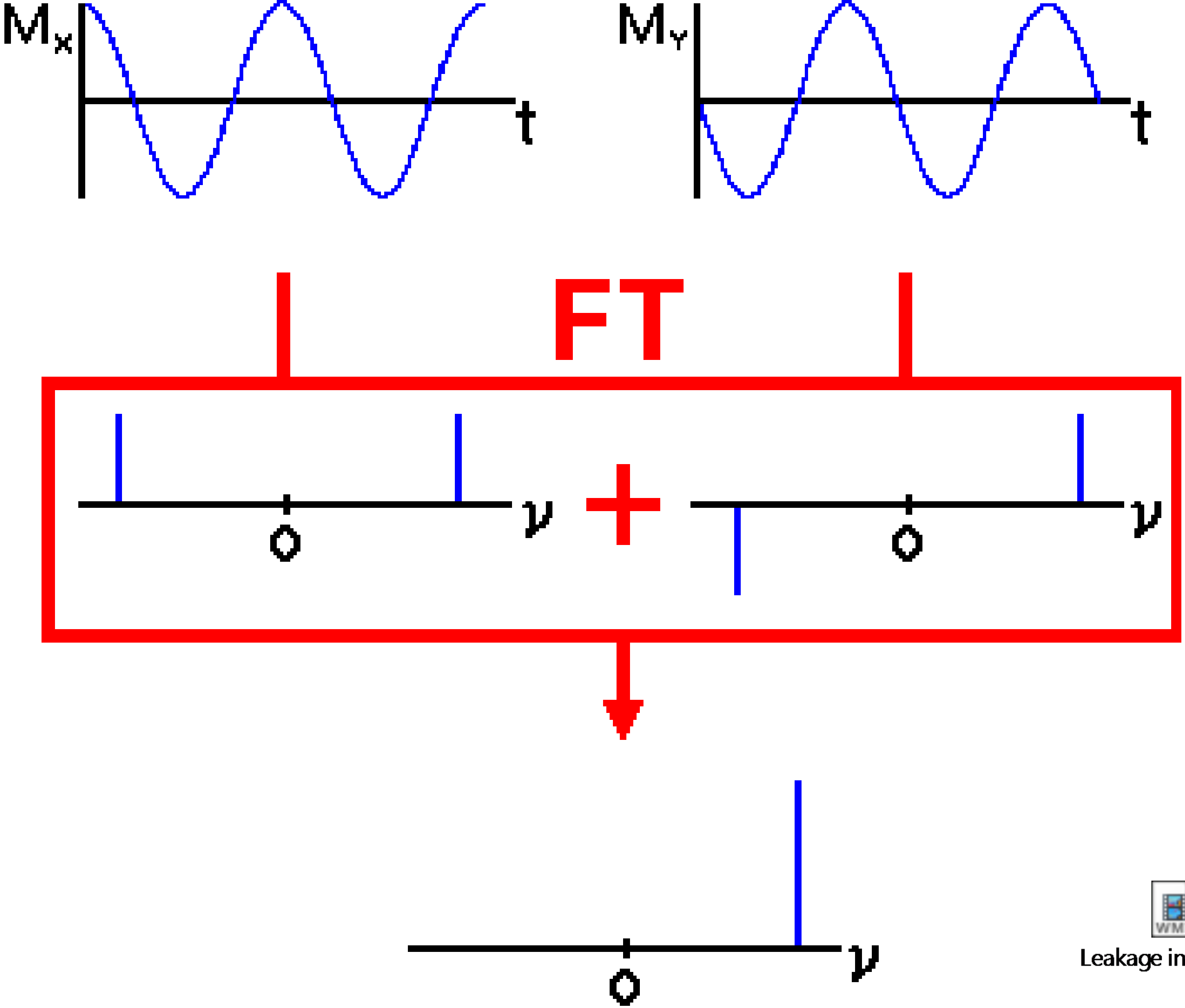
$$\delta(x) \leftrightarrow 1$$



$$\Pi(x) \leftrightarrow \mathrm{sinc}(u) = \frac{\sin(\pi\,\mathrm{u})}{\pi\,\mathrm{u}}$$

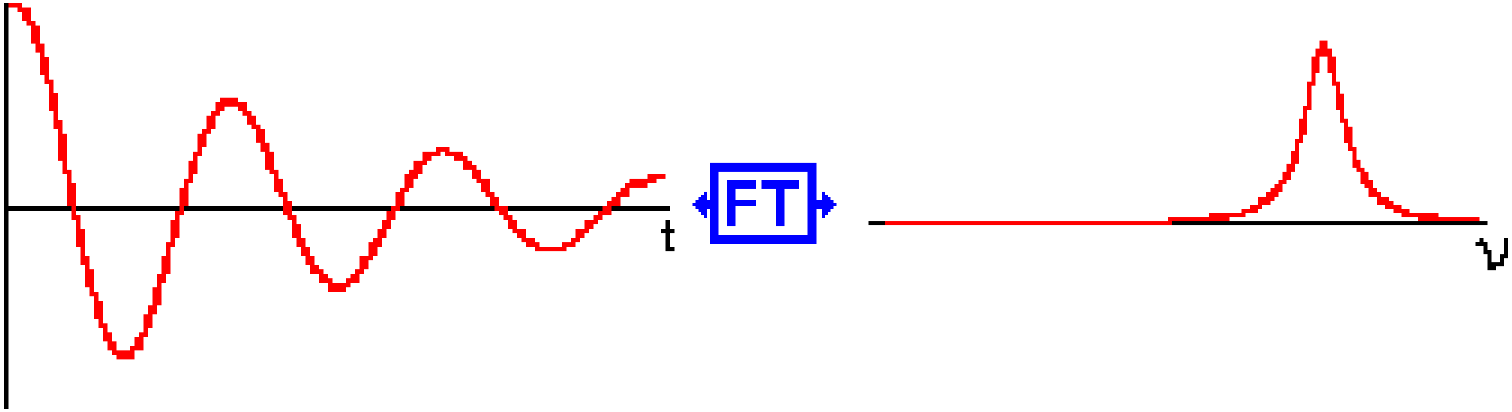
$$\left.\begin{matrix} \Lambda(x) & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{matrix}\right\} \leftrightarrow \mathrm{sinc}^2(u)$$



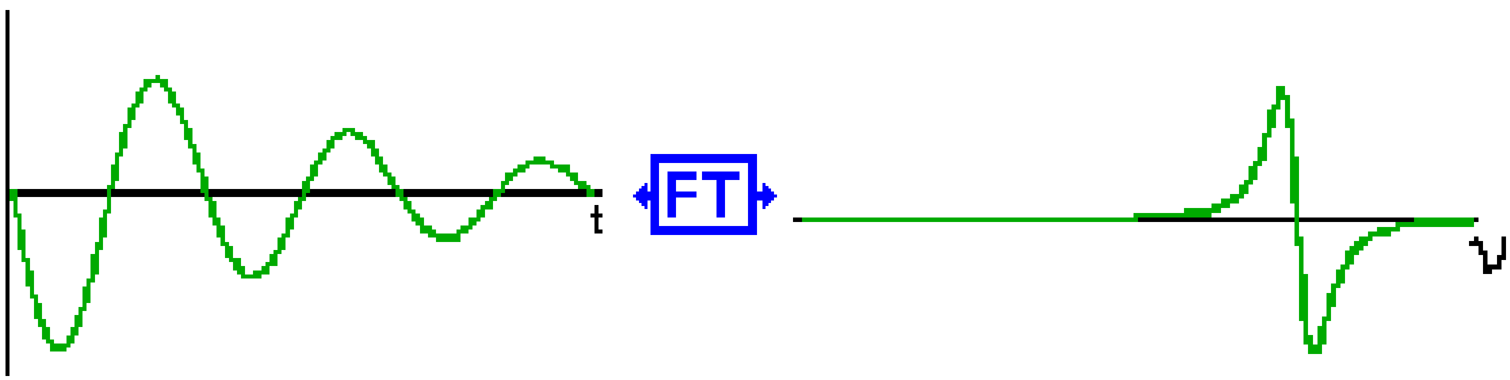


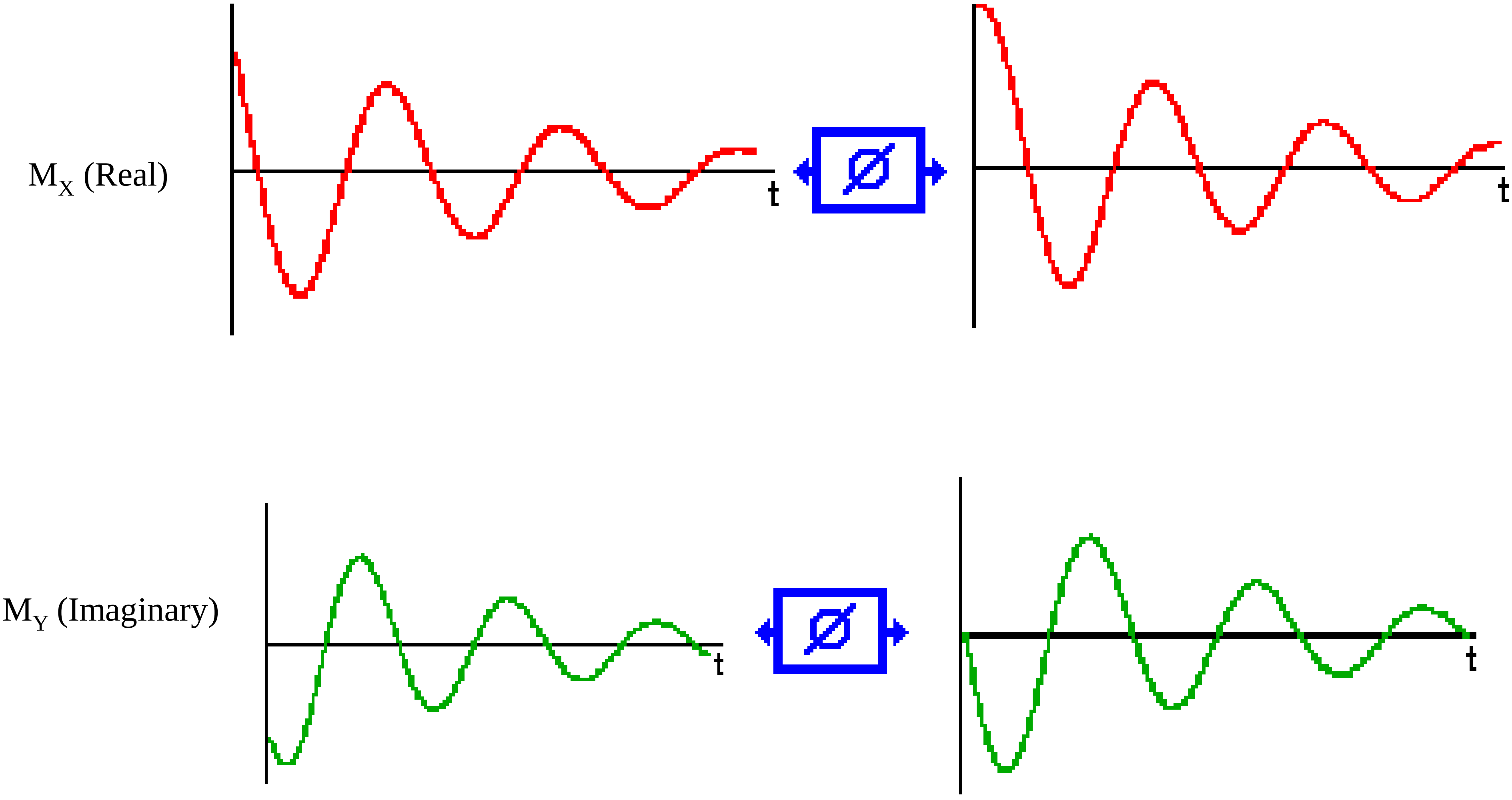
Leakage in FT.wmv

M_X (Real)



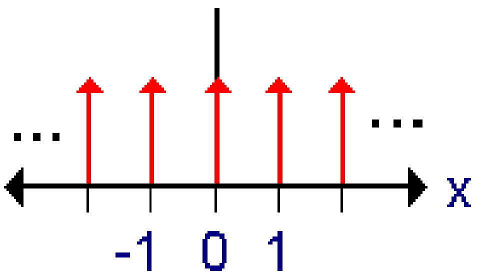
M_Y (Imaginary)




M_X (Real)
t
t
M_Y (Imaginary)
t
t

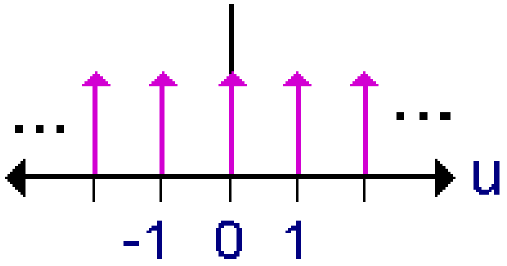
Some Fourier Transform Pairs and Definitions, continued

$$\mathrm{f}(x) \leftrightarrow \mathrm{F}(\mathrm{u})$$

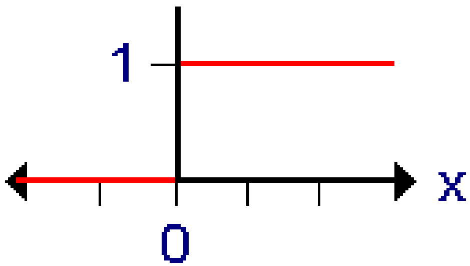


$$\mathrm{comb}(x) \text{ or } ⊔⊔(x) \leftrightarrow \mathrm{comb}(u)$$

$$= \sum_{n=-\infty}^{\infty} \delta(x-n)$$



$$H(x) = \begin{Bmatrix} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{Bmatrix} \leftrightarrow \tfrac{1}{2}\delta(u) - \frac{i}{2\pi u}$$



$$\delta(x\text{-}x_o) \leftrightarrow e^{-i\cdot 2\pi\cdot x_o\cdot u}$$

1-D Fourier transform properties

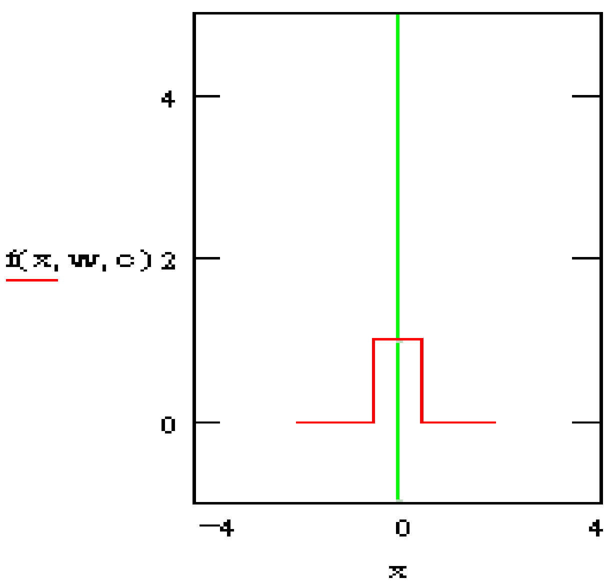
If $f(x) \leftrightarrow F(u)$ and $h(x) \leftrightarrow H(u)$,

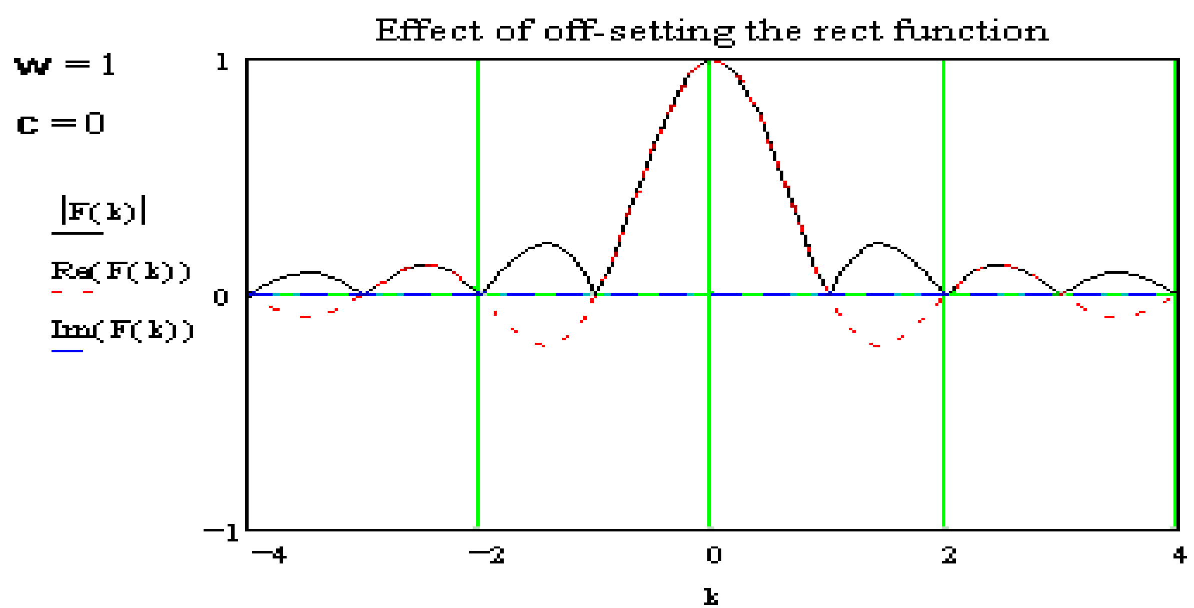
Linearity: $af(x) + bh(x) \leftrightarrow aF(u) + bH(u)$

Scaling: $f(ax) \leftrightarrow \frac{1}{|a|} F\left(\frac{u}{a}\right)$

Fourier Scaling theorem.avi

Shift: $f(x\text{-}a) \leftrightarrow F(u)e^{-i2\pi au}$





1-D Fourier transform properties, continued.

Say g(x) $\leftrightarrow$ G(u). Then,

Derivative Theorem: $$g'(x) \rightarrow i\,2\pi\,u\,G(u)$$

(Emphasizes higher frequencies – high pass filter)

Integral Theorem: $$\int_{-\infty}^{x} g(x)dx \rightarrow \frac{1}{2}G(u)\delta(u) + \frac{1}{i\,2\pi\,u}G(u)$$

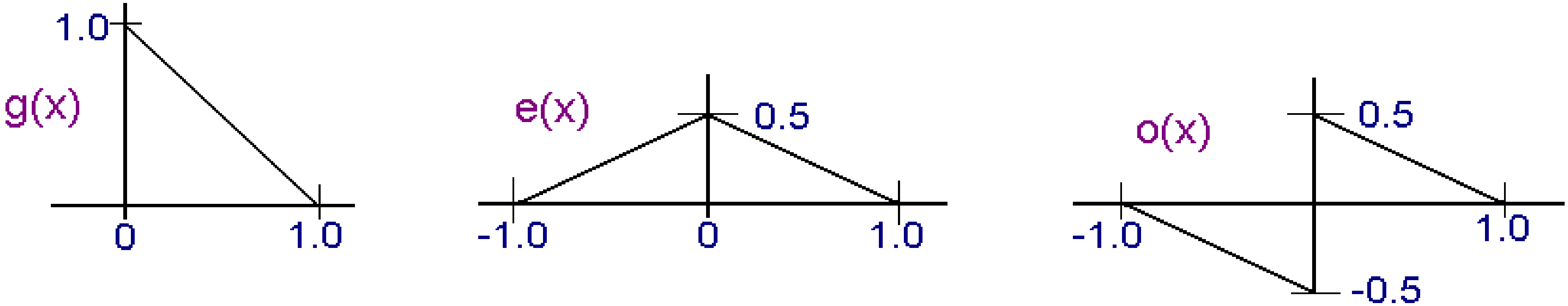
(Emphasizes lower frequencies – low pass filter)

Even and odd functions and Fourier transforms

Any function g(*x*) can be uniquely decomposed into an even and odd function.

$$e(x) = \tfrac{1}{2}(\, g(x) + g(-x)\,) \qquad o(x) = \tfrac{1}{2}(\, g(x) - g(-x)\,)$$

For example,



$$\int_{-\infty}^{\infty} e(x)dx = 2\int_{0}^{\infty} e(x)dx \qquad \int_{-\infty}^{\infty} o(x)dx = 0$$

$e_1 + e_2$ = even

$o_1 + o_2$ = odd

$e_1 \cdot e_2$ = even

$o_1 \cdot o_2$ = even

$e_1 \cdot o_1$ = odd

Fourier transforms of even and odd functions

Consider the Fourier transforms of even and odd functions.

$$\mathrm{g}(x) = \mathrm{e}(x) + \mathrm{o}(x)$$

$$\mathrm{G}(u) = \int_{-\infty}^{\infty}\left(\mathrm{e}(x)+\mathrm{o}(x)\right)\cos(2\pi\cdot u\cdot x)dx - i\int_{-\infty}^{\infty}\left(\mathrm{e}(x)+\mathrm{o}(x)\right)\sin(2\pi\cdot u\cdot x)dx$$

$$= 2\int_{0}^{\infty}\mathrm{e}(x)\cos(2\pi\cdot u\cdot x)dx - i\cdot 2\int_{0}^{\infty}\mathrm{o}(x)\sin(2\pi\cdot u\cdot x)dx$$

$$= \mathrm{E}(u) - i\mathrm{O}(u)$$

Sidebar:

E(u) and O(u) can both be complex if e(x) and o(x) are complex.

If g(x) is even, then G(u) is even.

If g(x) is odd, then G(u) is odd.

Special Cases

For a real-valued g(x) (e(x) , o(x) are both real),

$$\mathrm{Re}\{G(u)\} = \int_{-\infty}^{\infty} \mathrm{e}(x)\cos(2\pi ux)\,dx$$

$$\mathrm{Im}\{G(u)\} = -\int_{-\infty}^{\infty} \mathrm{o}(x)\sin(2\pi ux)\,dx$$

Real part is even in u

$$\mathrm{Re}\{G(u)\} = \mathrm{Re}\{G(-u)\}$$

Imaginary part is odd in u

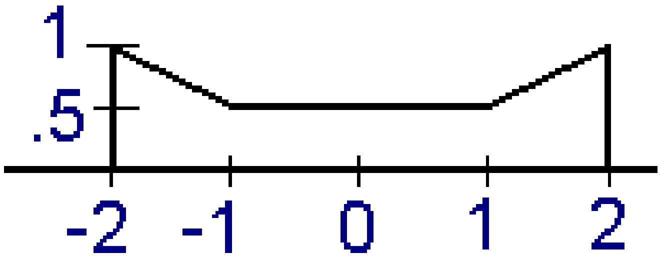
$$\mathrm{Im}\{G(u)\} = -\mathrm{Im}\{G(-u)\}$$

So, G(u) = G*(-u), which
is the definition of

Hermitian Symmetry: G(u) = G*(-u) (even in magnitude, odd in phase)

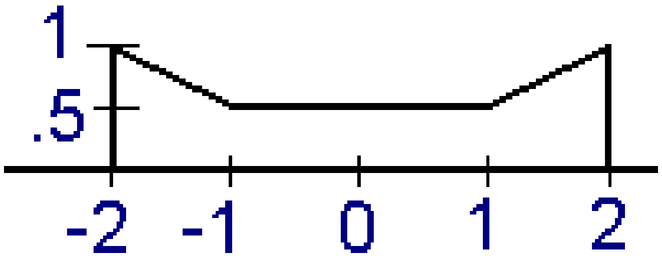
Example problem

Find the Fourier transform of

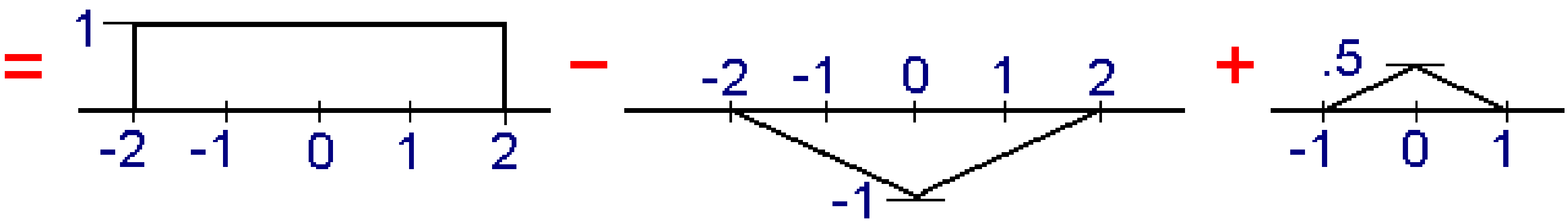


Example problem: Answer.

Find the Fourier transform of



$$f(x) = \Pi(x/4) - \Lambda(x/2) + .5\Lambda(x)$$

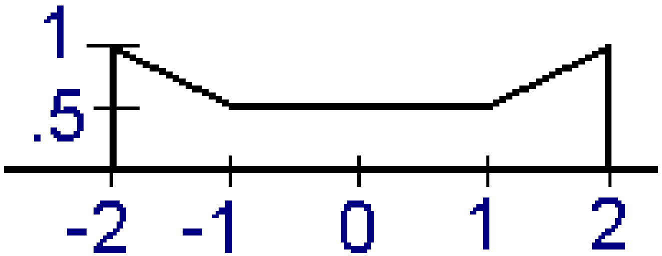


Using the Fourier transforms of Π and Λ and the linearity and scaling properties,

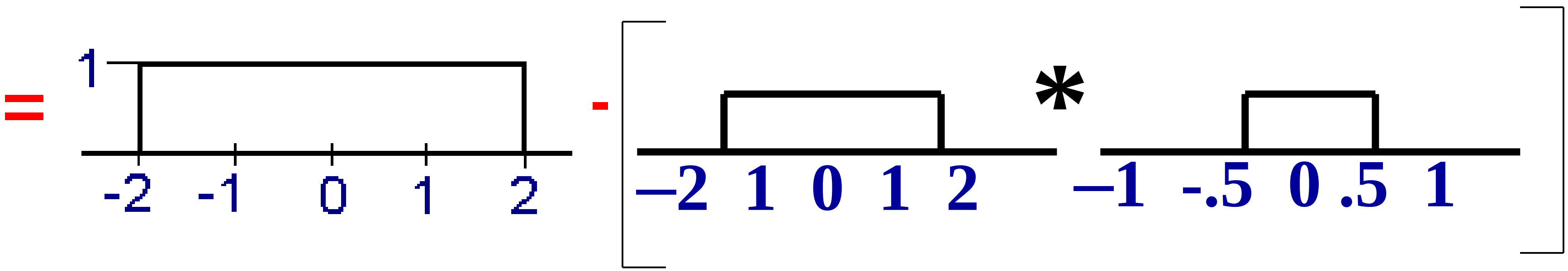
$$F(u) = 4\text{sinc}(4u) - 2\text{sinc}^2(2u) + .5\text{sinc}^2(u)$$

Example problem: Alternative Answer.

Find the Fourier transform of



$$f(x) = \Pi(x/4) - 0.5((\Pi(x/3) * \Pi(x))$$



Using the Fourier transforms of Π and Λ

and the linearity and scaling and convolution properties ,

$$F(u) = 4\mathrm{sinc}(4u) - 1.5\mathrm{sinc}(3u)\mathrm{sinc}(u)$$

2D Fourier Transfrom

$$f\left(U,V\right)=\sum_{x=1}^{N}\sum_{y=0}^{M}f\left(x,y\right)e^{-2\pi i\left(Ux+Vy\right)}$$

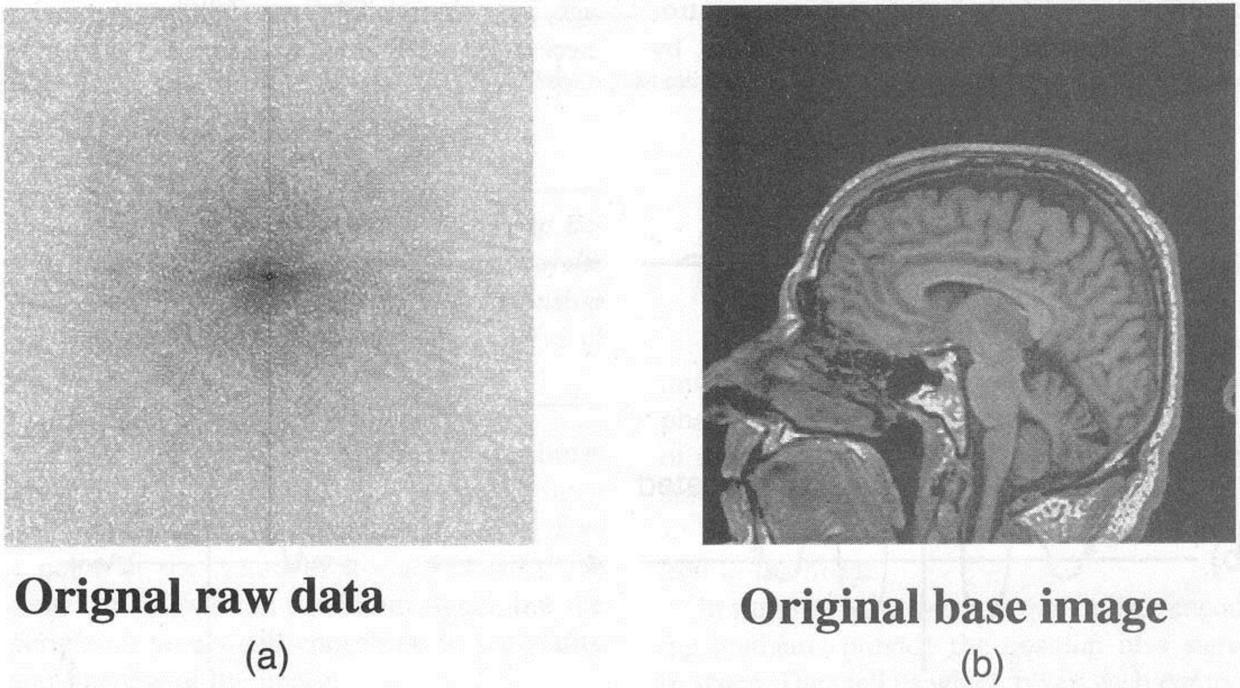
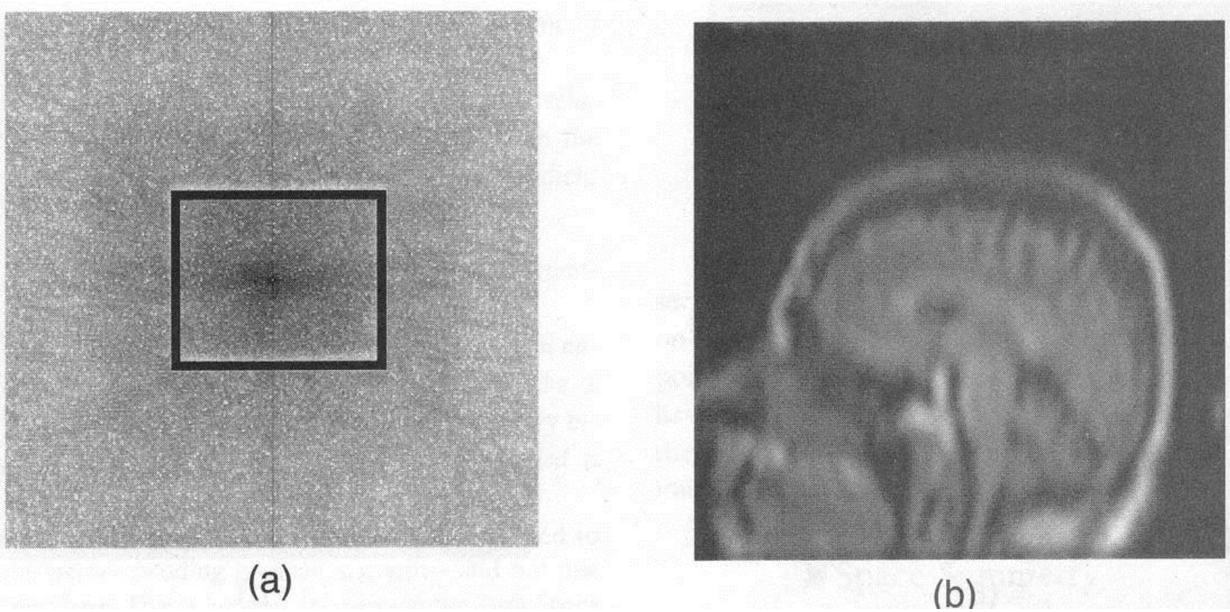
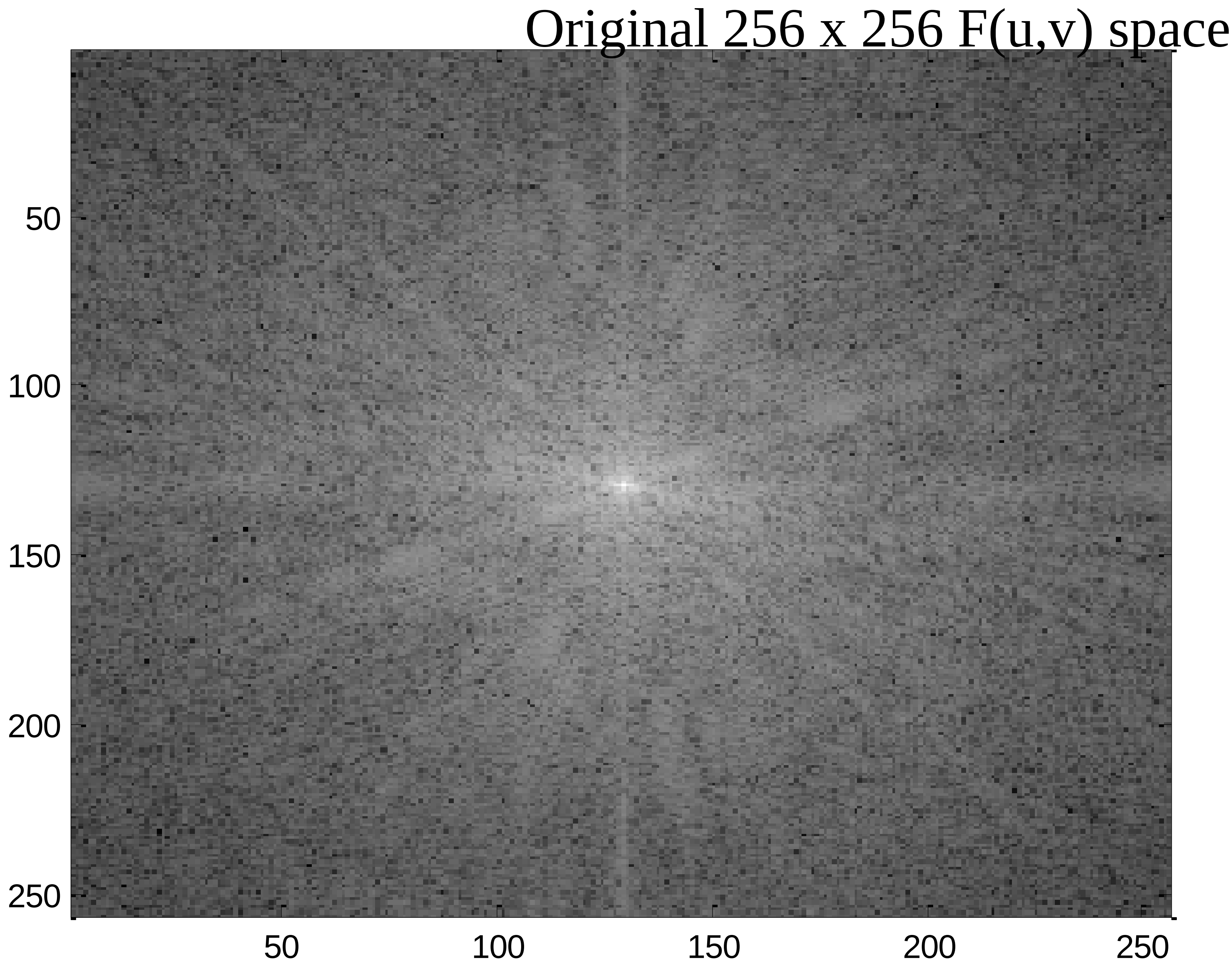
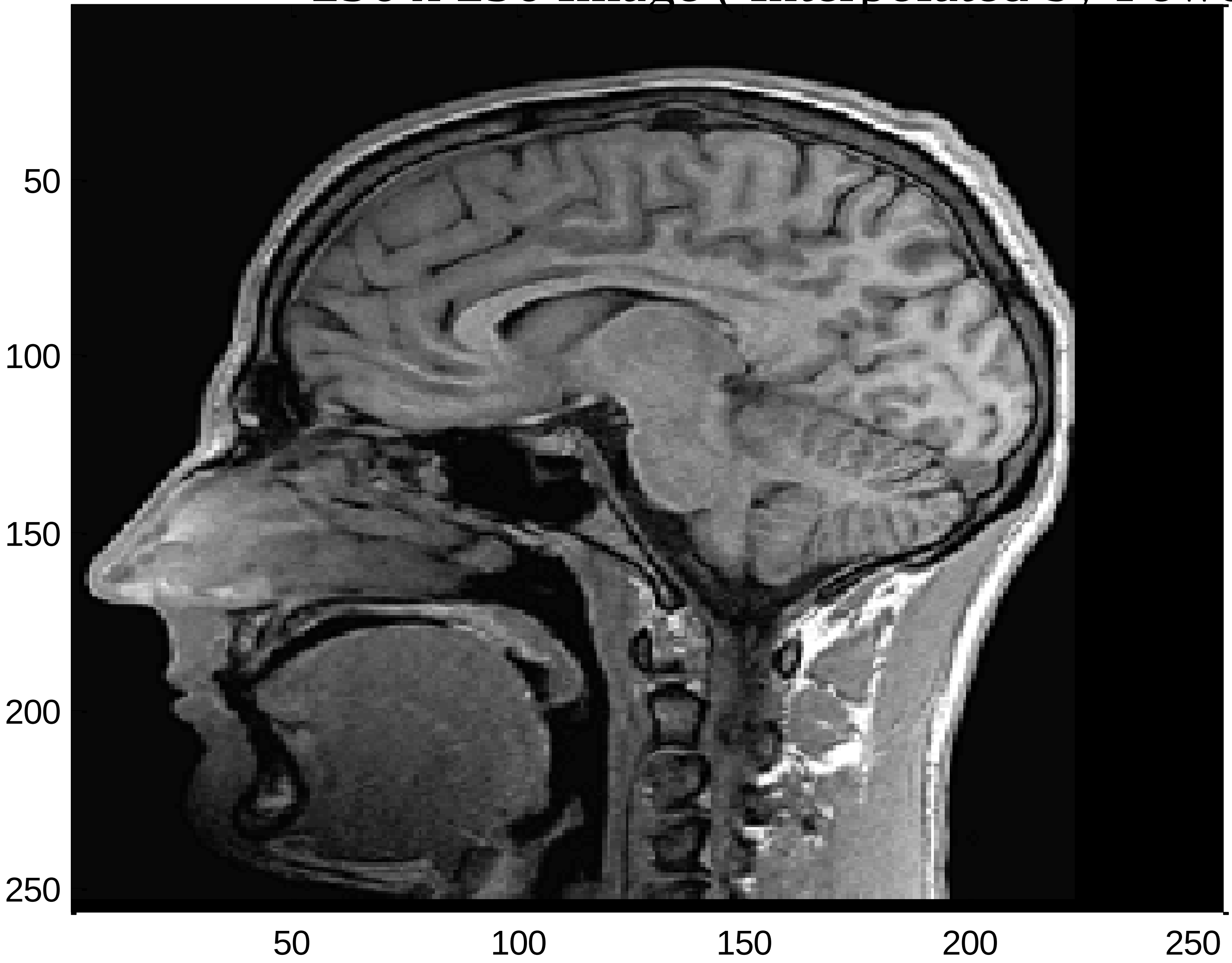


Figure 13-5. **(a)** The original raw data (k-space) of **(b)** the original image (midline sagittal T_1- weighted image of the brain).

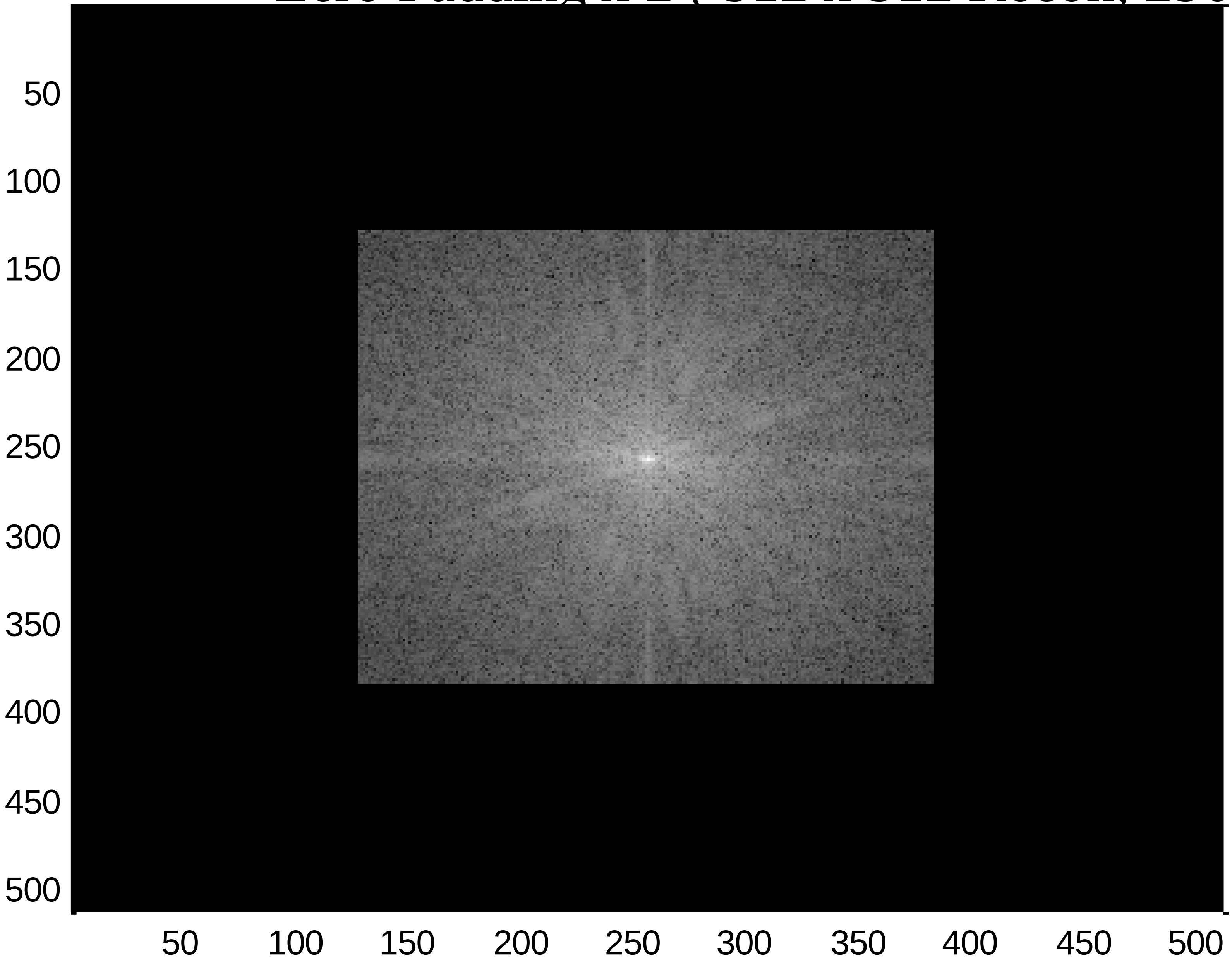



Original 256 x 256 F(u,v) space
50
100
150
200
250
50
100
150
200
250

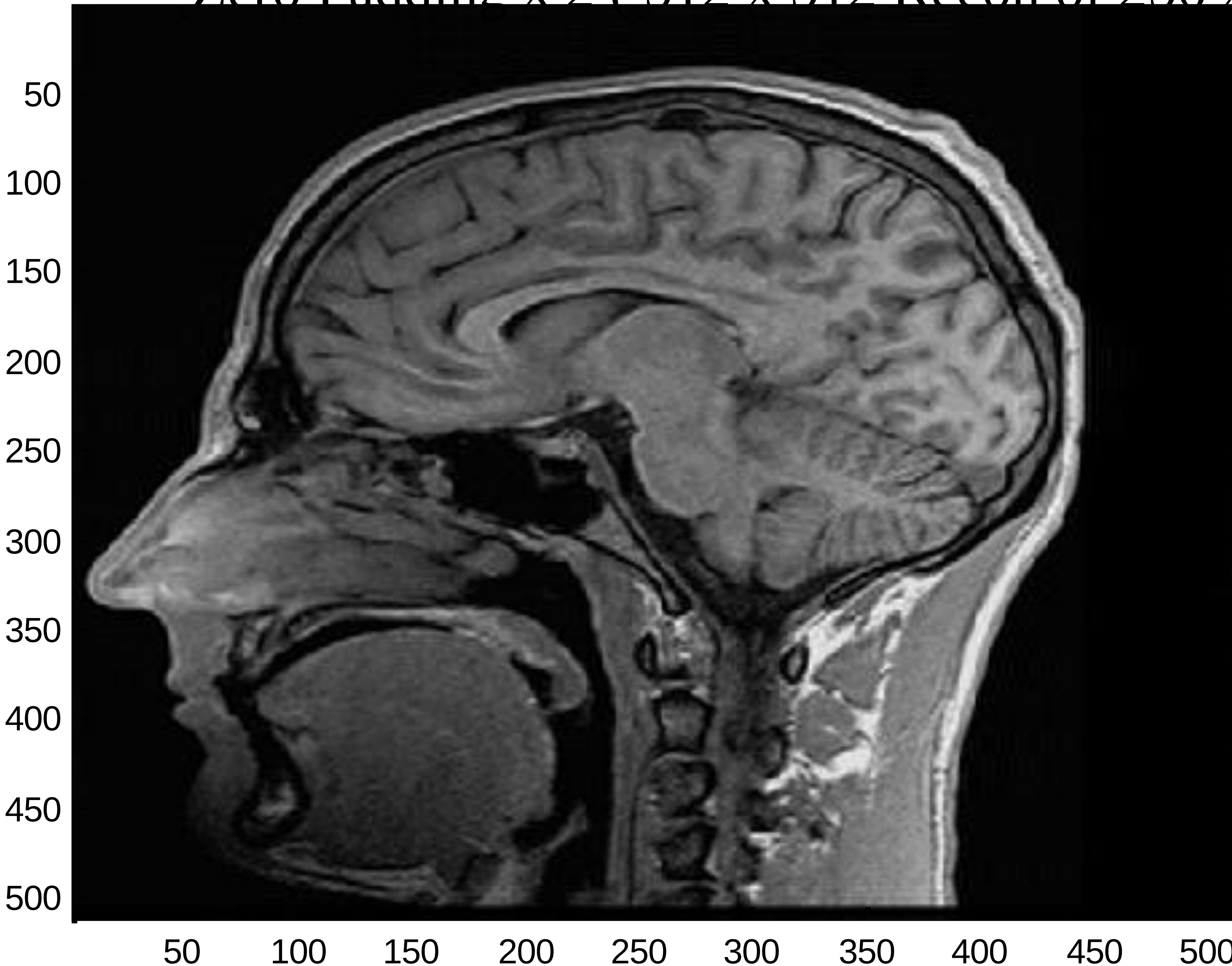
256 x 256 Image (Interpolated by Powerpoin



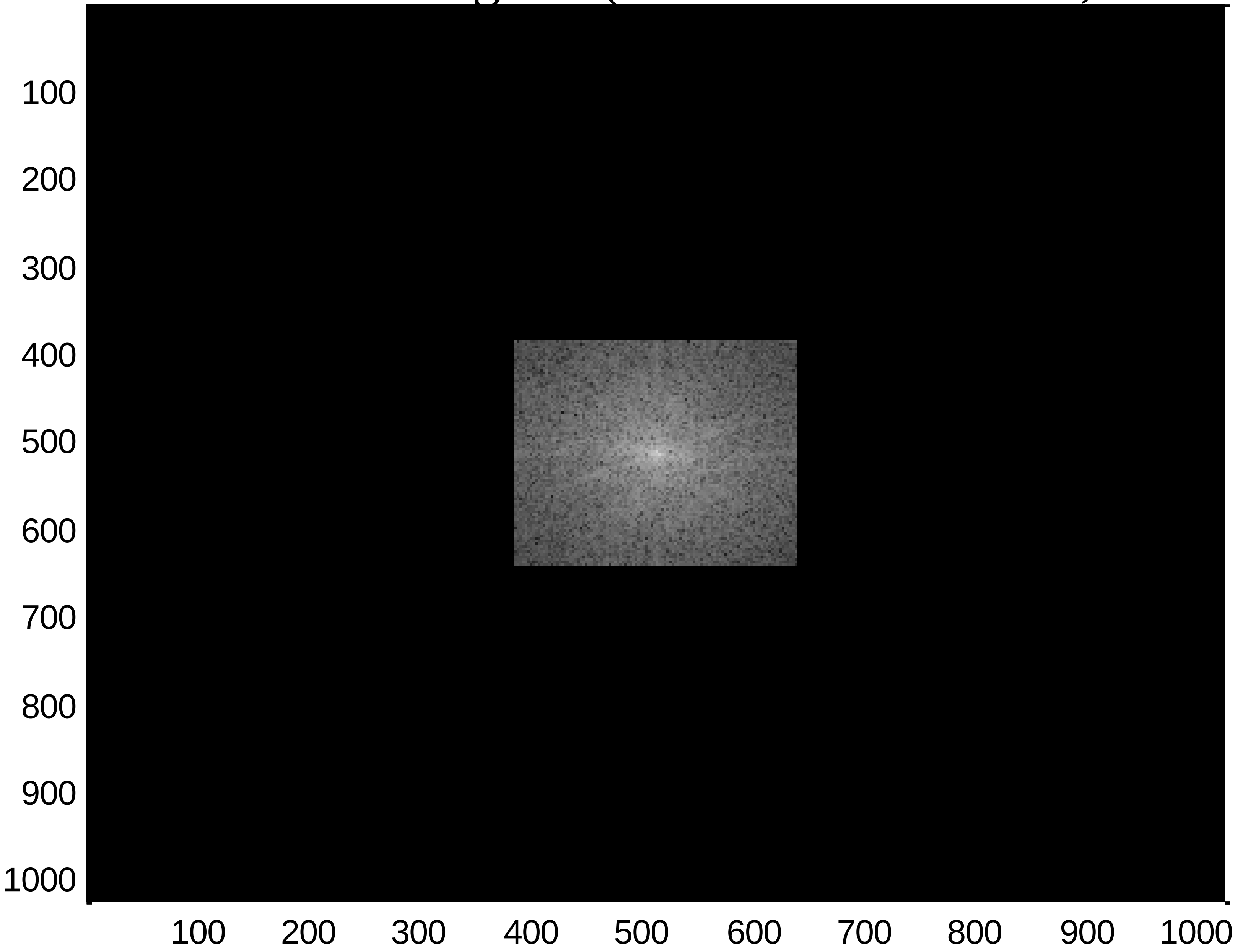
Zero Padding x 2 (512 x 512 Recon, 256 x 256



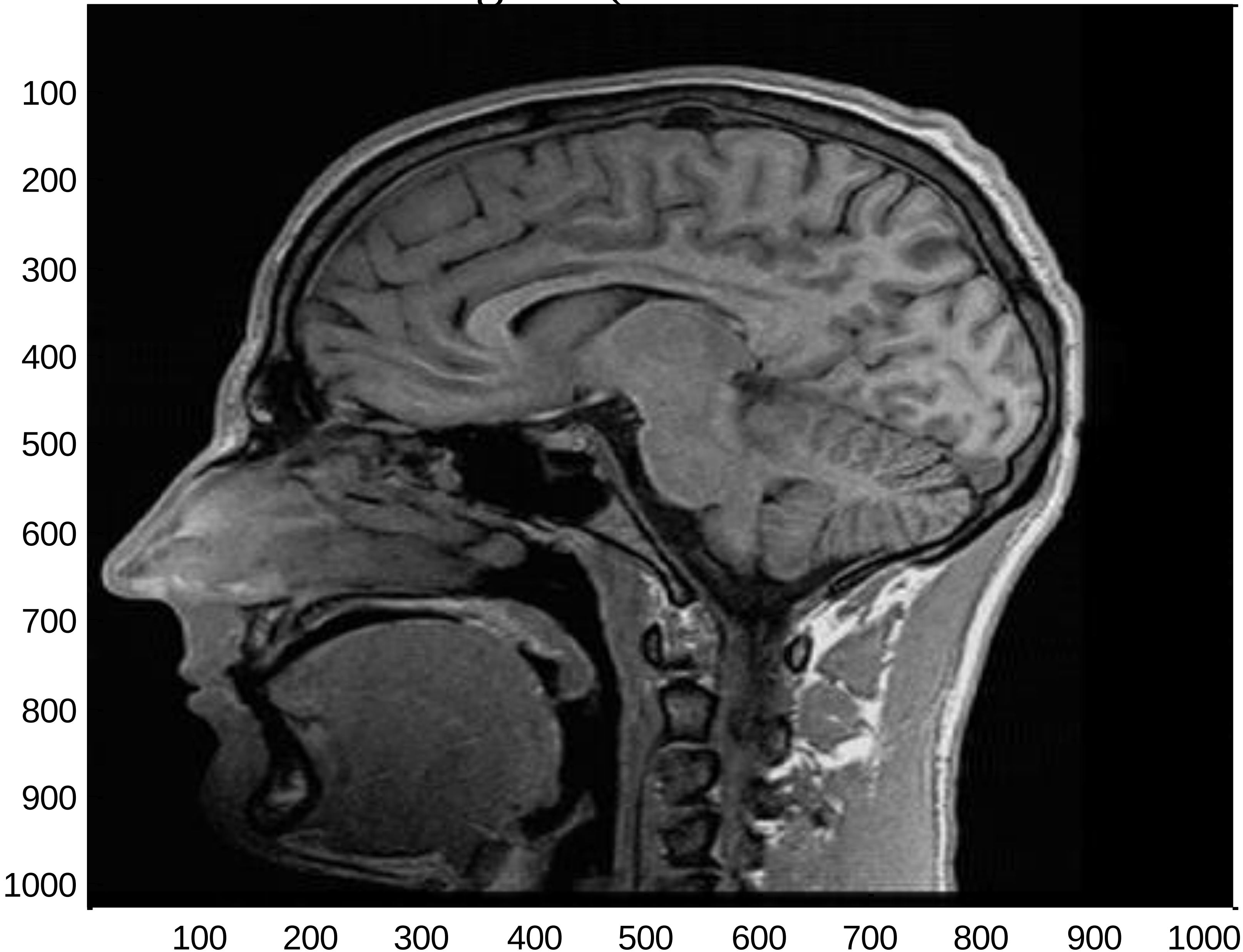
Zero Padding x 2 (512 x 512 Recon of 256 x 256 I



Zero Padding x 4 (1024 x 1024 Recon, 256 x 256 K



Zero Padding x 4 (1024 x 1024 Recon of 256 x 256



Discrete Fourier Transform

- تبدیل فوریه در مسا ئل طبیعي و عادي به صورت يك تخمين تحليلي رياضي (Analytic expression) بكار مي رود.
- به اين تخمين، *تبديل فوريه گسسته* (Discrete Fourier Transform) مي گويند.

- DFT، نسخه نمونه برداري شده FT (در گستره بي نهايت) است كه از طريق تكرار (replication) بخشي از آن در محدوده نمونه برداري شده بدست مي آيد. لذا اطلاعات در فضاي فركانسي بصورت مجموعه ء گسسته نقاط در نظر گرفته مي شود.

$$Xj = jDx, \quad j=0,1,2,\ldots.,N-1$$

$$Un = nDU, \quad n=0,1,2,\ldots.,M-1$$

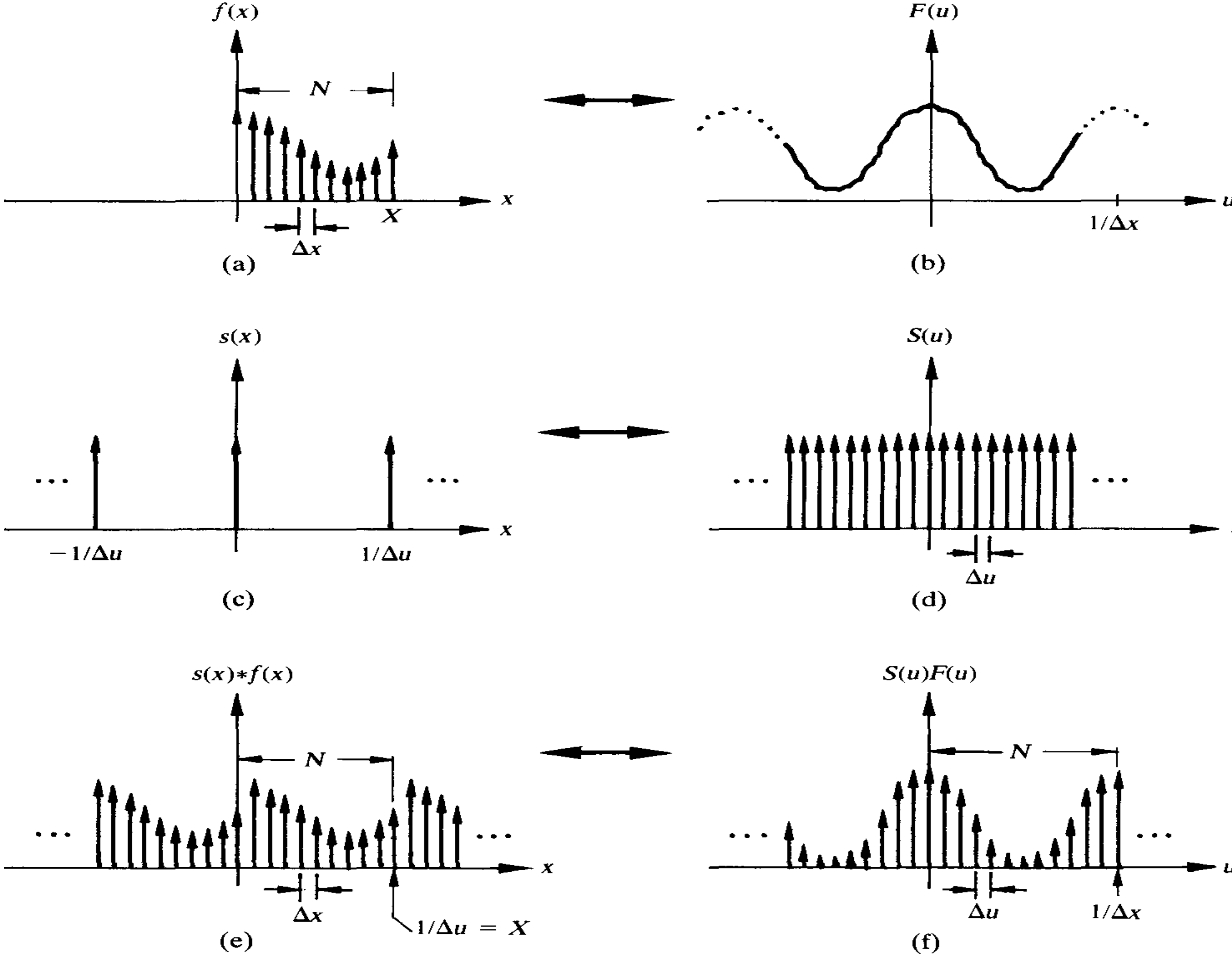
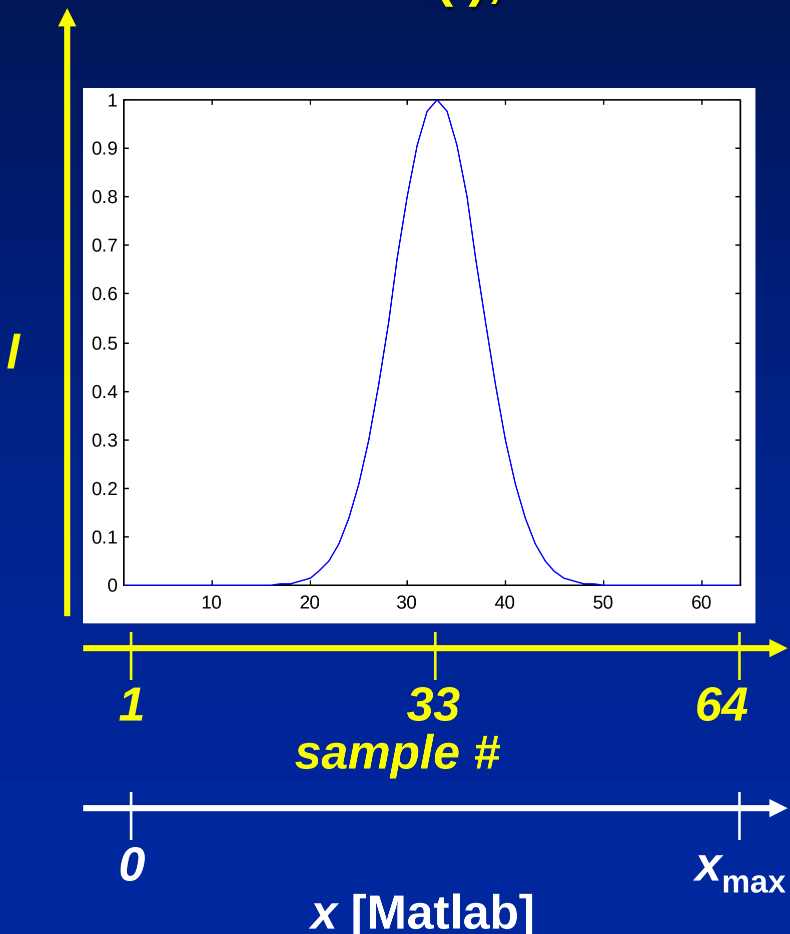


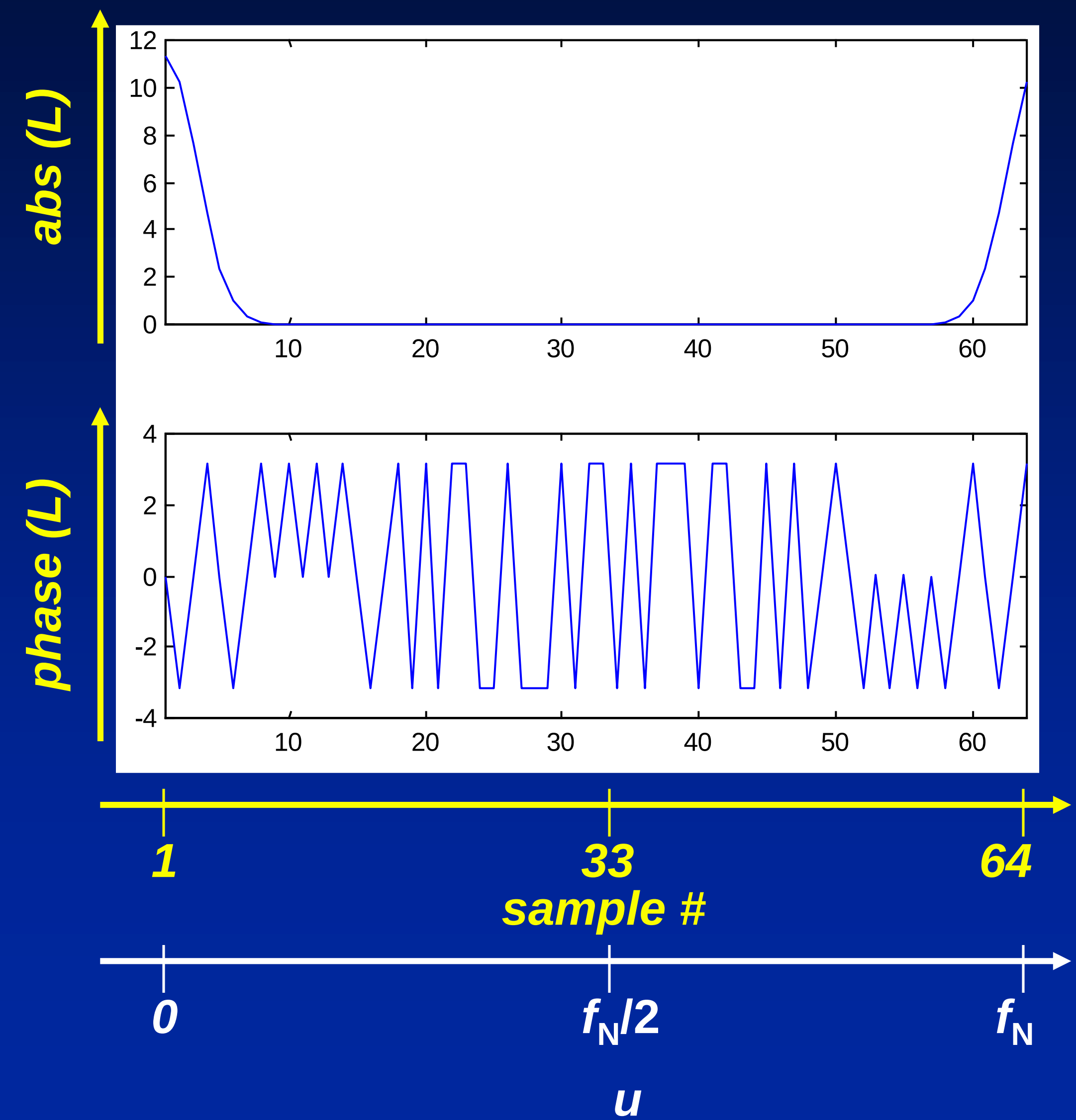
Figure 3.19 *Graphic illustration of the discrete Fourier transform.*

FT Conventions - 1D in Matlab

1D Fourier Transform

- `>>l = z(33,:);`
- `>>L = fft(l);`

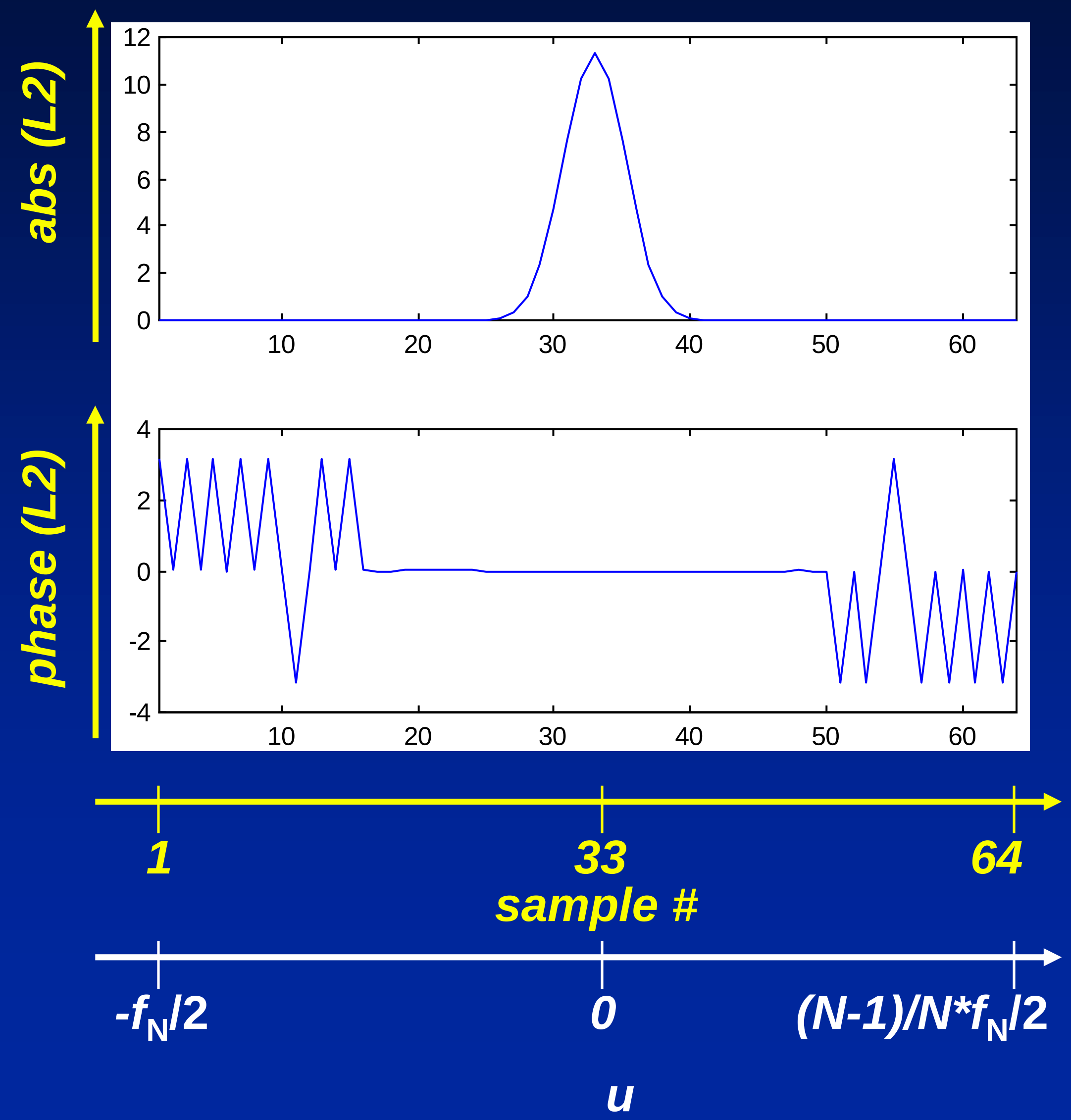




1D FT - 2xfftshift

Center Fourier Domain

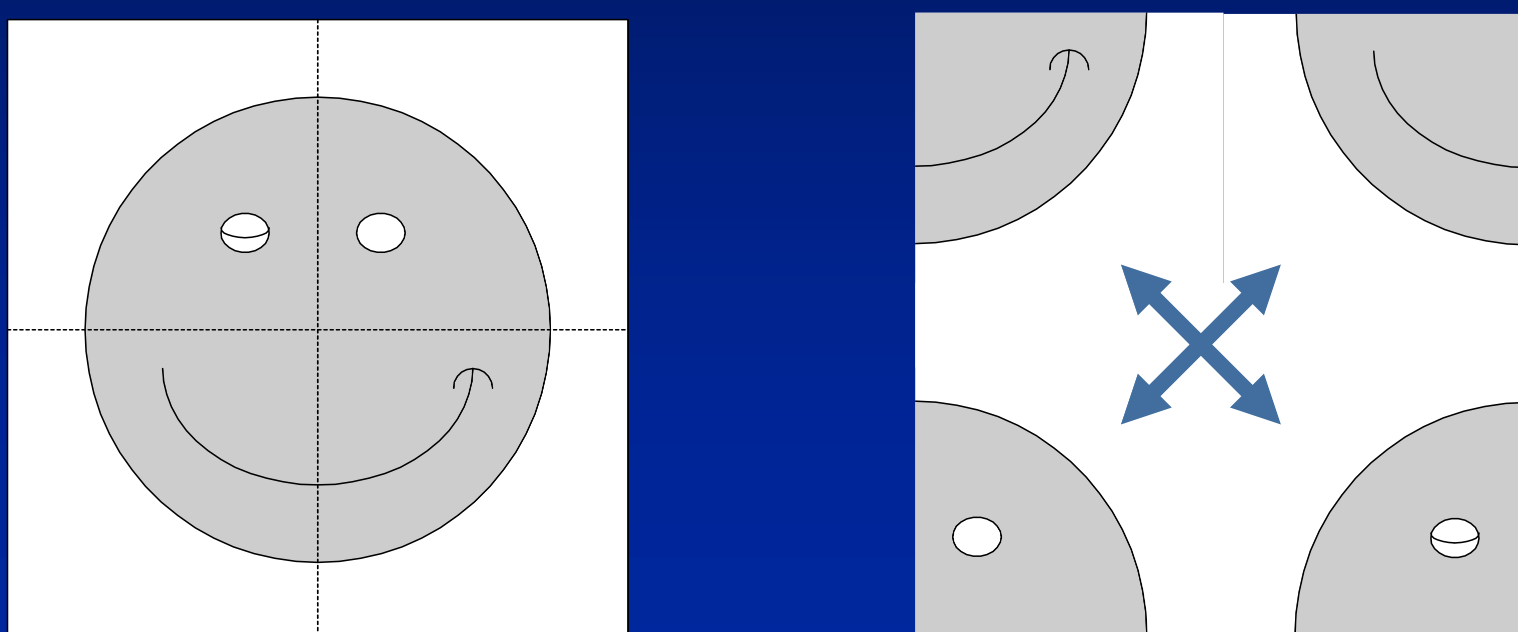
- >>L2 =
  fftshift(fft(I));



fftshift in 2D

Use fftshift for 2D functions

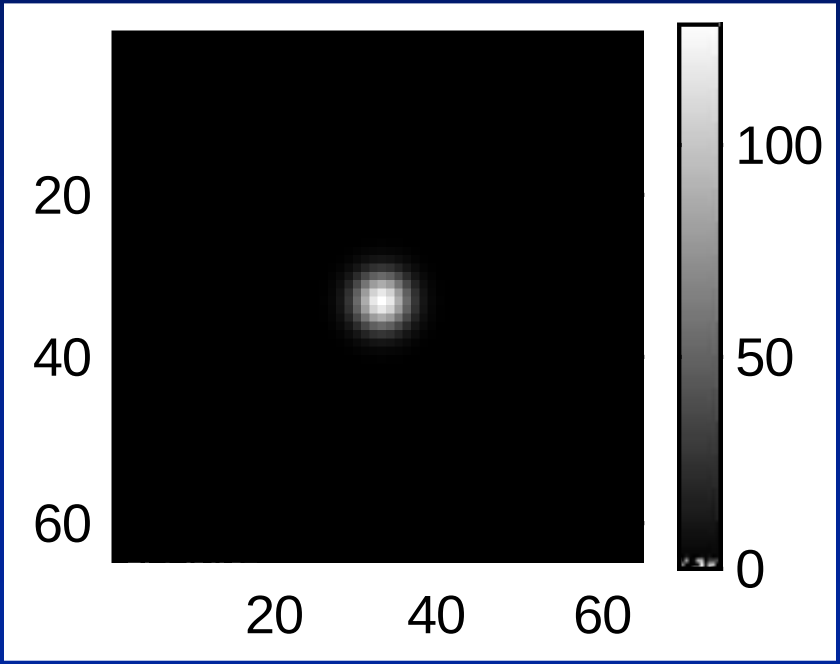
- >>smiley2 = fftshift(smiley);



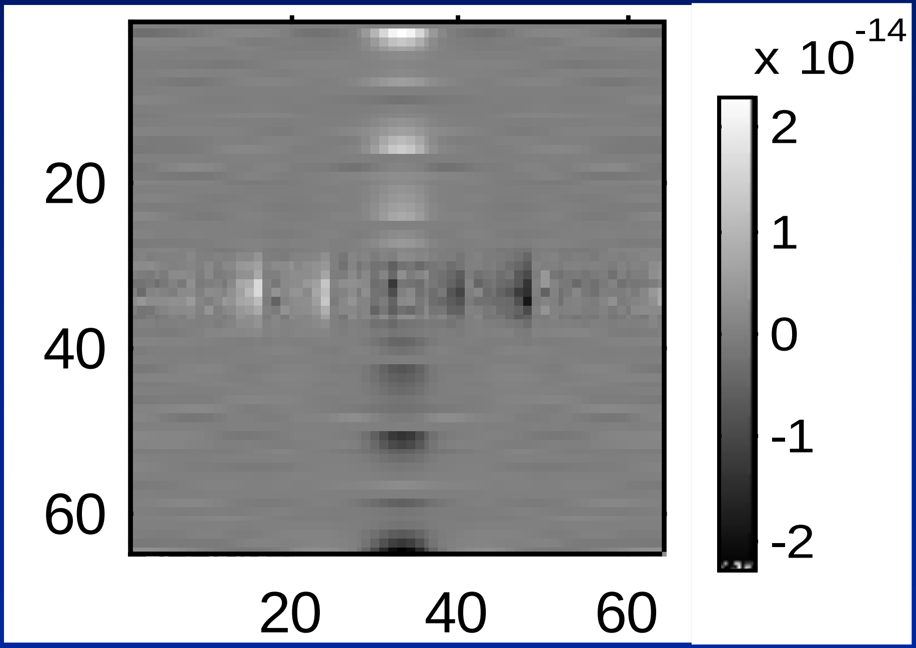
2D Discrete Fourier Transform

- >>Z=fftshift(fft2(z));
- >>whos
- >>figure
- >>imshow(real(Z))
- >>imshow(real(Z),[])
- >>colorbar

- >>figure
- >>imshow(imag(Z),[])
- >>colorbar



Real (Z)



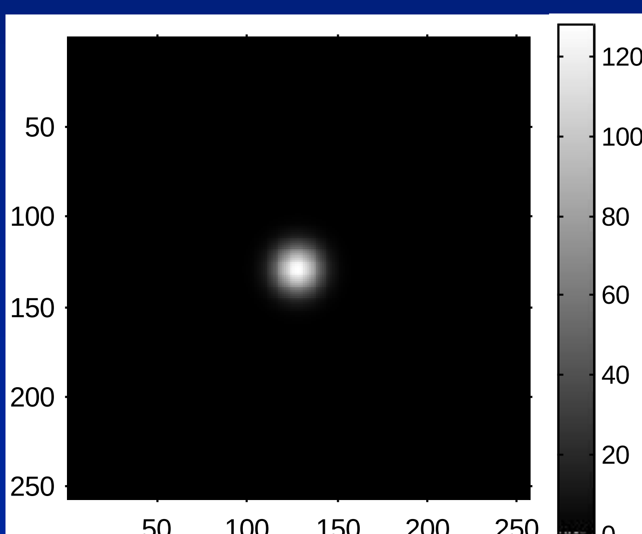
Imag (Z)

2D DFT - Zero Filling

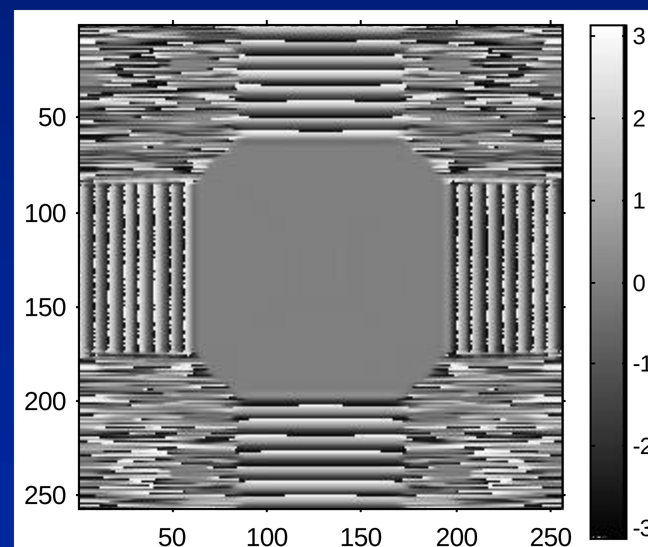
Interpolating Fourier space by zerofilling in image space

- >>z_zf = zeros(256);
- >>z_zf(97:160,97:160) = z;
- >>Z_ZF = fftshift(fft2(z_zf));
- >>figure
- >>imshow(abs(Z_ZF),[])
- >>colorbar

- >>figure
- >>imshow(angle(Z_ZF),[])
- >>colorbar



Magnitude (Z_ZF)



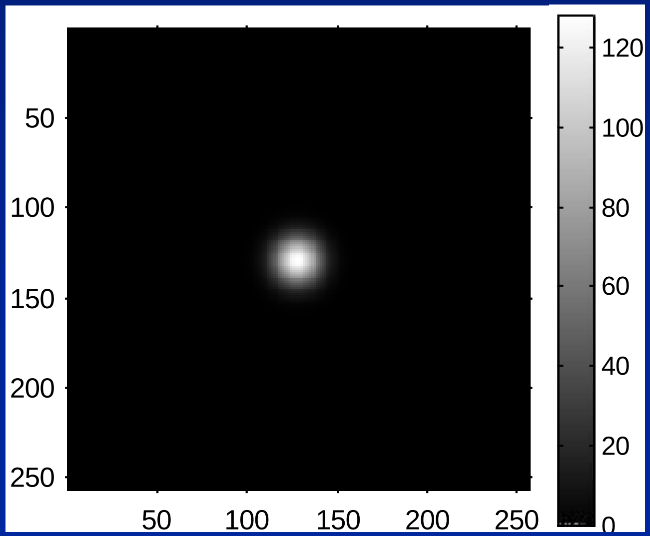
Phase (Z_ZF)

2D DFT - Voxel Shifts

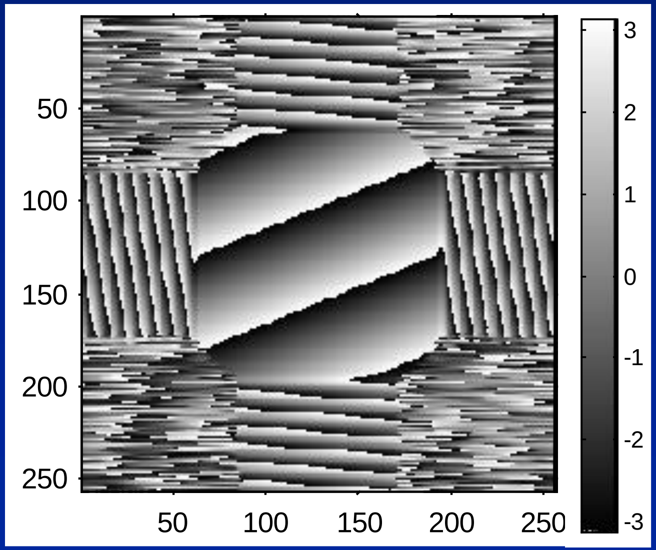
Shift image: 5 voxels up and 2 voxels left

- >>z_s = zeros(256);
- >>z_s(92:155,95:158) = z;
- >>Z_S = fftshift(fft2(z_s));
- >>figure
- >>imshow(abs(Z_S),[])
- >>colorbar

- >>figure
- >>imshow(angle(Z_S),[ ])
- >>colorbar



Magnitude (Z_S)



Phase (Z_S)