Mathematics of Diffusion Tensor Imaging

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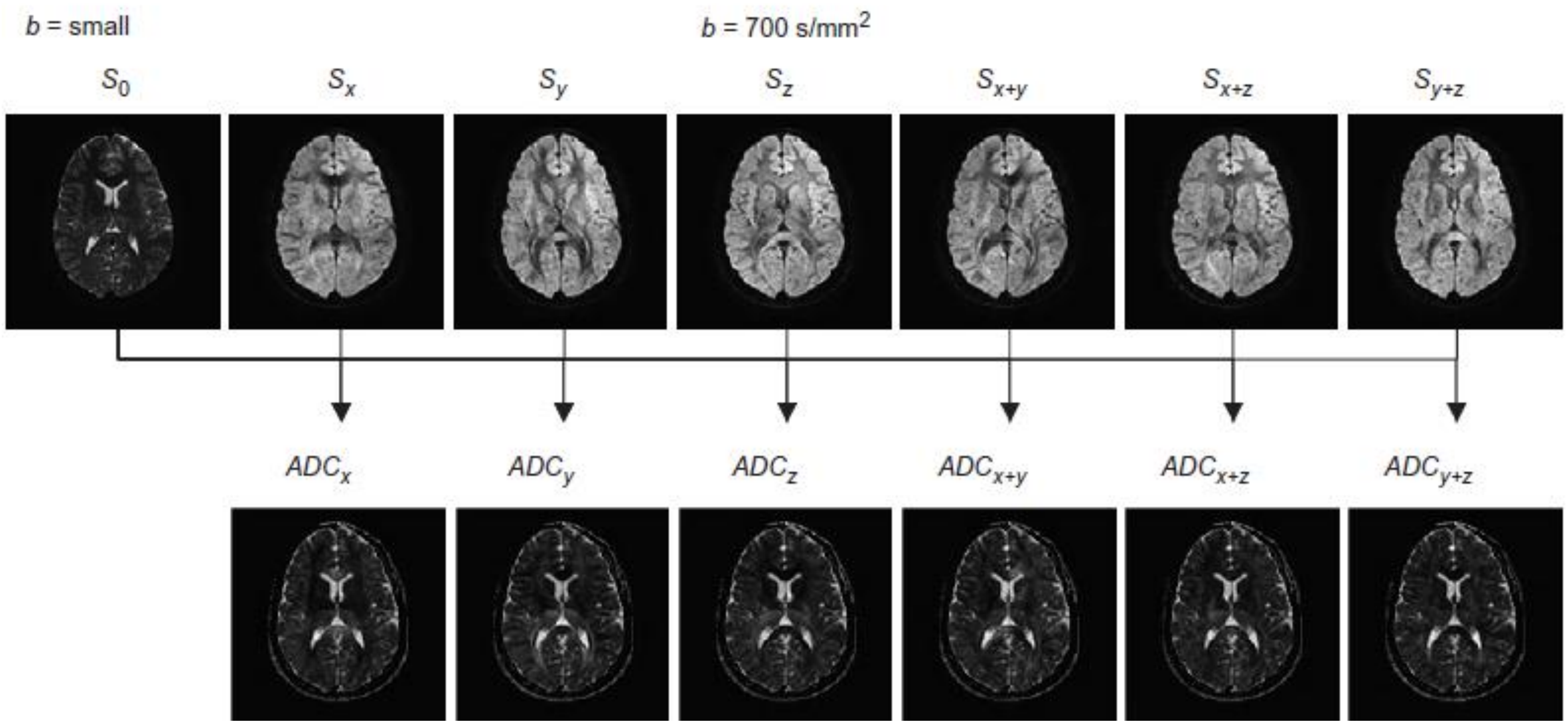
Algorithm to measure diffusion ellipsoid from diffusion measurements along six independent axes

- At least two data points are needed to obtain a diffusion constant from a slope of signal attenuation.
- This include one low-diffusion-weighted image corresponds to b0, and one of 6 main combined directions in x, y, and z.
- Apparent Diffusion Constant along each direction, eg; in x-gradient (Sx), we can calculate an the x-axis (ADCx).
- 6 ADCs are obtained using various gradient combinations x, y, z, x+y, x+z, y+z gradients

$$\frac{S}{S_0} = e^{-\gamma^2 G^2 \delta^2 (\Delta - \delta/3) D} = e^{-bD}$$

A b0 (S0 Image) and six DWI images along six different axes are needed



For 6 gradient direction

- For a directional gradients, we obtain:

$$\frac{S}{S_0} = e^{-b\overline{g}^T \overline{\overline{\mathbf{D}}}\,\overline{g}}$$

- b is $\gamma^2 |G^2 \delta^2 \ (\Delta - \delta/3)$
- $\overline{g}$ is a unit vector pointing along the direction of the gradient

<u>In actual experiments, the six parameters in the matrix (or tensor) D are what we want to determine</u>

Equation $\frac{S}{S_0} = e^{-b\bar{g}^T \bar{\bar{\mathbf{D}}} \bar{g}}$ has a total of 6 unknowns D but we have at least 7 experimental results (S) with different g and b values, Therefore, it can be solved.

- For example if we use the x-gradient, $\bar{g}$ [1,0,0], measure image intensity S_x, and put this number into above Eq., the $e^{-b\bar{g}^T \bar{\bar{\mathbf{D}}} \bar{g}}$ portion becomes

$$\gamma^2 G^2 \delta^2 (\Delta - \delta/3)\ (1,0,0) \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- If we solve this part and put it back into previous Eq., it becomes:

$$\frac{S_x}{S_0} = e^{-\gamma^2 G^2 \delta^2 (\Delta - \delta/3) D_{xx}}$$

from which we can obtain the element D_{xx}.

- Similarly, from experiments using the y- and z-gradient only, we can obtain D_{yy} and D_{zz}.

- If we apply the same strength to (x+y)gradient simultaneously, we obtain the image intensity S_{xy}

$$\bar{g} = [\sqrt{1/2}, \sqrt{1/2}, 0$$

$$\frac{S_{xy}}{S_0} = e^{-\gamma^2 G^2 \delta^2 (\Delta - \delta/3)(1/2D_{xx} + D_{xy} + 1/2D_{yy})}$$

Because we already know D_{xx} and D_{yy}, we can calculate D_{xy} from this result.

Similarly D_{xz} and D_{yz} can be obtained

TO DETERMINE THE 6 PARAMETERS OF A DIFFUSION ELLIPSOID

- The 6 elements of D are used and converted to give:
  - To obtain these 6 parameters from 3*3*3 diffusion tensor, $\overline{\overline{D}}$ a process called "diagonalization" if used.

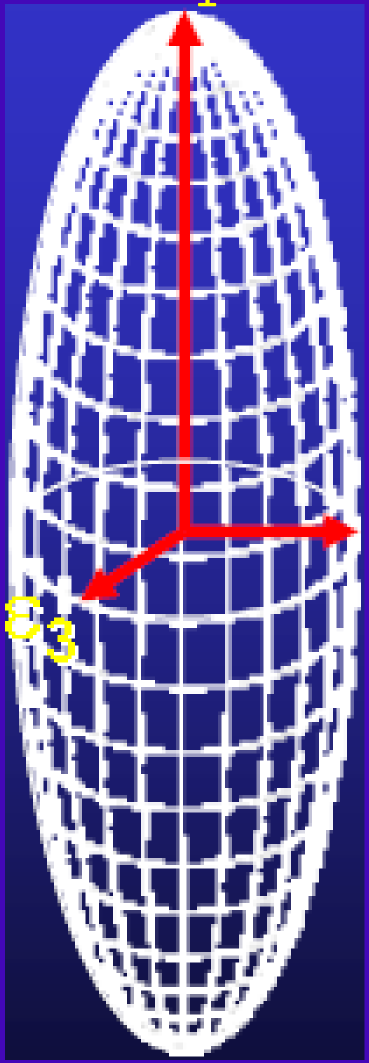
$$\overline{\overline{\mathbf{D}}} = \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix} \xrightarrow{\textit{diagonalization}} \lambda_1,\ \lambda_2,\ \lambda_3,\ \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$$

3 numbers for the length of the longest (λ1), middle (λ2), and shortest (λ3) axes, which define the ellipsoid shape

3 vectors (v1, v2, v3) to define ellipsoid orientations (principal axes).

Eigenvectors

ε_1- Principal Eigenvector



Eigenvalues:

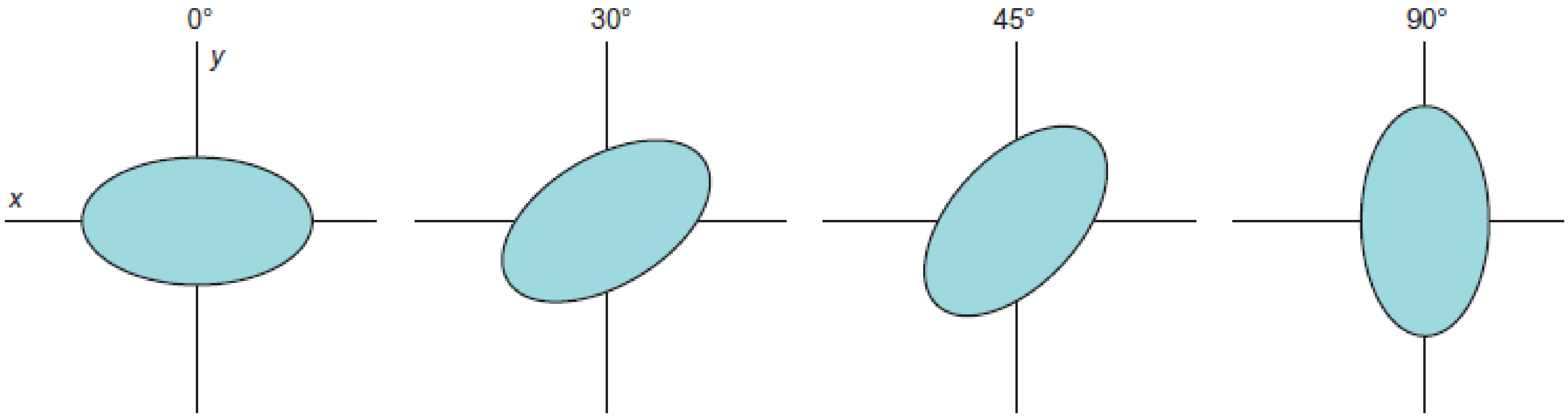
λ_1- Largest Eigenvalue

λ_2- Intermediate Eigenvalue

λ_3 – Smallest Eigenvalue

The three eigenvectors are *always* perpendicular to each other

6 PARAMETERS OF DIFFUSION ELLIPSOID vs diffusion matrix D



	0°	30°	45°	90°
D	$\begin{bmatrix} 2.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 1.75 & -0.43 & 0 \\ -0.43 & 1.25 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 1.5 & -0.5 & 0 \\ -0.5 & 1.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 2.0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$
λ_1	2.0	2.0	2.0	2.0
λ_2	1.0	1.0	1.0	1.0
λ_3	0.5	0.5	0.5	0.5
v_1	[1, 0, 0]	[0.87, 0.5, 0]	[0.71, 0.71, 0]	[0, 1, 0]
v_2	[0, 1, 0]	[-0.5, 0.87, 0]	[-0.71, 0.71, 0]	[1, 0, 0]
v_3	[0, 0, 1]	[0, 0, 1]	[0, 0, 1]	[0, 0, 1]
ADC_x	2.0	1.75	1.5	1.0
ADC_y	1.0	1.25	1.5	2.0
ADC_z	0.5	0.5	0.5	0.5
Trace	3.5	3.5	3.5	3.5

How to measure diffusion tensor D from more gradient direction

- For more gradient direction, a fitting technique should be used rather than solving it.
- We need to expand the Eq. to

$$b\bar{g}^T \overline{\overline{\mathbf{D}}}\bar{g} = b[\, g_x \quad g_y \quad g_z \,] \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix} \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}$$

$$= b(D_{xx}g_x^2 + D_{yy}g_y^2 + D_{zz}g_z^2 + 2D_{xy}g_xg_y + 2D_{xz}g_xg_z + 2D_{yz}g_yg_z)$$

$$= \bar{b} \cdot \bar{D}$$

- Here, we have defined two new vectors, $\overline{b}$ and $\overline{D}$ which are defined as:

$$\begin{aligned} \overline{b} &= b[g_x^2,\ g_y^2,\ g_z^2,\ 2g_xg_y,\ 2g_xg_z,\ 2g_yg_z] \\ \overline{D} &= [D_{xx},\ D_{yy},\ D_{zz},\ D_{xy},\ D_{xz},\ D_{yz}] \end{aligned}$$

Then the main equation can be rewritten to:

$$\ln(S) = \ln(S_0) - \overline{b} \cdot \overline{D}$$

- This equation is very similar to a simple linear equation, **y=constant-ax**,
- **y** is the measurement (equivalent to ln(S)),
- **ln(S0)** is the constant
- **a** is the slope (equivalent to the b vector),
- **x** is the independent variable (here D vector).

- This equation can be solved by **linear least-square fitting**, to obtain D

Example of solving Diffusion tensor using least square fitting

- for one 0-gradient and 6 gradient combination of:
  [0,0,0], [1,0,0], [0,1,0], [0,0,1], $\left[\sqrt{2},\sqrt{2},0\right],\left[\sqrt{2},0,\sqrt{2}\right],\left[0,\sqrt{2},\sqrt{2}\right]$

(This correspond to 7 b-vectors $(\overline{b_1},\overline{b}_2,......,\overline{b}_7)$
, each having 6 parameters,

And image intensities S1, S2,. . .,S7

From each measurement; then, the entire experiment can be expressed as:

In which $\overline{\overline{\mathrm{B}}}$ is called the b-matrix.

Eg: with 30 different direction $\overline{b}$(i=1,2, . . ., 31)

then 31 images (S_1, . . ., S_{31}) are obtained (1 for 0-gradeint and 30 directions).

In this equation, best estimated $\overline{D}$ can be obtained using multivariate linear fitting.

$$\begin{bmatrix} \ln(S_1) \\ \ln(S_2) \\ \ln(S_3) \\ \ln(S_4) \\ \ln(S_5) \\ \ln(S_6) \\ \ln(S_7) \end{bmatrix} = \ln(S_0) - \begin{bmatrix} \overline{b}_1 \\ \overline{b}_2 \\ \overline{b}_3 \\ \overline{b}_4 \\ \overline{b}_5 \\ \overline{b}_6 \\ \overline{b}_7 \end{bmatrix} \cdot \overline{D}$$

$$= \ln(S_0) - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 1 \end{bmatrix} \cdot \overline{D}$$

$$\rightarrow \overline{S} = \ln(S_0) - \overline{\overline{\mathbf{B}}}\overline{D}$$